5. Professor Stouffer showed that an easy method of introducing point-duality into projective geometry is by considering it as the space dual of plane-duality. Such a method is in line with the desire to form the duals of every concept introduced. Numerous illustrations of the different kinds of duality were shown.

6. Professor Stratton presented briefly the results of the freshmen engineering tests in arithmetic and algebra which had been conducted under the direction of Dr. J. C. Peterson of the Department of Education of the Kansas State Agricultural College. While the tests have not been carried far enough to justify any definite conclusions, yet there seems to be a close correlation between the test grades and the ability of the students to do their college work successfully.

H. E. JORDAN, *Secretary-Treasurer.*

THE APRIL MEETING OF THE KENTUCKY SECTION.

The fourth annual meeting of the Kentucky Section was held at Centre College, Danville, Kentucky, on April 17; Professor C. G. CROOKS of Centre College was chairman of the meeting.

There were ten present including the following seven members of the Association; P. P. Boyd, C. G. Crooks, J. M. Davis, H. H. Downing, E. L. Rees, C. H. Richardson, G. W. Smith. The section was the guest of the retiring chairman at an excellent dinner. For the ensuing year Professor H. H. DOWNING was elected chairman, and Dr. G. W. SMITH, secretary-treasurer.

The following six papers were read;

(1) "Calculus ideas before Newton and Leibnitz," by Professor P. P. BOYD;

(2) "Some problems concerning the catenary," by Professor H. H. DOWNING;

(3) "A solution of Euler's equation," by Professor C. H. RICHARDSON;

(4) "Some functions of a single nilpotent number," by Dr. G. W. SMITH;

(5) "An early treatment of the quadratic transformation," by J. OSBORN, Graduate Student, University of Kentucky. (By invitation.)

(6) "A hodge-podge of mathematical nonsense," by Professor E. L. REESE.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. Professor Boyd outlined the calculus ideas of Archimedes, Kepler, Cavalieri, Pascal, Descartes, Fermat, Barrow, and Roberval and presented demonstrations from the works of these men by way of illustration. Finally Newton's and Leibnitz's treatments of the problem of tangents and quadratures were given.

2. Professor Downing made brief mention of the general theory in which it is shown that if $y = f(x)$ is the equation of a curve joining two fixed points and $I = \int F(x, y, y')dx$, then for I to be a minimum (or maximum) $f(x)$ must be a solution of the Euler differential equations. Also that the Weierstrass condition, the Legendre condition and the Jacobi condition relating to conjugate points, must be satisfied by $f(x)$.

For the integral, $I = \int_a^b y\sqrt{1 + y'^2}\,dx$, which is the integral appearing in the expression for the surface obtained by revolving the curve $y = f(x)$ about the X-axis, he pointed out that the catenary is the solution of the Euler equations. The other conditions were applied and the results discussed. MacNeish's discussion of the Goldschmidt discontinuous solution was presented.

3. Professor Richardson proved the following theorem analytically, and found thereby a solution of Euler's equation

$$\frac{dt'}{\sqrt{R(t)}} + \frac{dt''}{\sqrt{R(t)}} = 0 \tag{1}$$

where $R(t) = At^4 + 2Bt^3 + Ct^2 + 2Dt + E$: *If* Σ *is a conic represented parametrically by the equations*

$$x = \frac{a_1t^2 + b_1t + c_1}{a_3t^2 + b_3t + c_3}, \qquad y = \frac{a_2t^2 + b_2t + c_2}{a_3t^2 + b_3t + c_3}$$

and if Σ' *is a second conic cutting* Σ *in four points of which the arguments are roots of an equation of the fourth degree* $R(t) = 0$, *any tangent whatever of* Σ' *intersects* Σ *in two points of which the arguments* t' *and* t'' *and their differentials when the tangent varies are connected by the relation* (1). Hence, in order to write a solution to this equation we find the condition that the line joining two points of Σ whose arguments are t' and t'' shall be tangent to Σ', the general conic which cuts Σ in four points whose arguments are roots of $R(t) = 0$. The relation obtained between t' and t'' contains an arbitrary constant and is a solution of the given equation.

4. After proving as does Peirce that in every linear associative algebra there is at least one idempotent or one nilpotent number Dr. Smith considered polynomials of a single nilpotent number j, $A(j) = \sum a_n j^n$ $(n = 0, 1, 2, 3, \cdots, \mu - 1)$ where $j^\mu = 0$ but $j^{\mu-1} \neq 0$. If $A(j) \cdot B(j) = 0$ and $a_0 \neq 0$, then $B(j)$ vanishes entirely. However if the first h of the coefficients of $A(j)$ are zero then the last h coefficients of $B(j)$ are arbitrary. $A(j)/B(j) = C(j)$ will exist (though part may be arbitrary) provided $A(j)$ does not start with a lower power of j than does $B(j)$. If $a_0 = a_1 = a_2 = \cdots = a_{kn-1} = 0$, but $a_{kn} \neq 0$ $(k = 0, 1, 2, \cdots)$ then $[A(j)]^n$ may be uniquely determined for any positive value of n.

5. Mr. Osborn briefly reviewed the first work done on quadratic transformations then discussed Steiner's transformation by means of the simple hyperboloid.

6. Professor Rees in a semi-humorous way pointed out some of the inconsistencies, the fallacies, and the improprieties of mathematics and their nomenclature, presenting thereby many of the things which so trouble the beginner.

GUY W. SMITH, *Secretary-Treasurer.*

FIFTH ANNUAL MEETING OF THE OHIO SECTION.

The fifth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, on the afternoon and evening of April 2, 1920, in connection with the meetings of the Ohio College Association. The widely advertised program attracted a large attendance not only of mathematicians, but also of physicists, other scientists and educators. The attendance included the following thirty-five members of the Association:

R. B. Allen, W. E. Anderson, G. N. Armstrong, C. L. Arnold, C. B. Austin, Grace M. Bareis, L. A. Bauer, W. S. Beckwith, R. D. Bohannan, R. L. Borger, W. D. Cairns, V. B. Caris, E. H. Clarke, O. L. Dustheimer, T. M. Focke, Harris Hancock, William Hoover, H. W. Kuhn, A. C. Lunn, C. N. Mills, C. N. Moore, C. C. Morris, A. D. Pitcher, J. B. Preston, S. E. Rasor, C. Lois Rea, Hortense Rickard, W. G. Simon, K. D. Swartzel, T. Elmer Trott, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, D. T. Wilson, and E. I. Yowell.

Professor R. L. BORGER occupied the chair, being relieved for an interval by Professor S. E. RASOR. Many of those in attendance at the afternoon meetings attended the dinner at the Ohio Union with the Ohio College Association, the dinner being followed by addresses of interest to college teachers. This gathering adjourned at 8.30 to attend Dr. Bauer's lecture of the evening program.

At the business meeting the Secretary reported that the membership of the Section is seventy, two more than last year, nine names having been removed from the roll and eleven added. Besides there are eight institutional members. The following officers were elected: Chairman, Professor S. E. RASOR, Ohio State University; Secretary-Treasurer, Professor G. N. ARMSTRONG, Ohio Wesleyan University; Third member of the executive committee, Professor A. D. PITCHER, Western Reserve University.

The following program was carried through:

(1) Address of the chairman: "Some geometric methods for curve tracing" by Professor R. L. BORGER;

(2) "The theory of relativity" by Professor A. C. LUNN;

(3) Symposium on Number (2) and related questions by (*a*) F. C. BLAKE, Professor of Physics, Ohio State University; (*b*) Dr. C. W. CHAMBERLAIN, President of Denison University; (*c*) D. C. MILLER, Professor of Physics, Case School of Applied Science; (*d*) Members of the Section and visitors.

(4) "The deflection of light observed during the solar eclipse of May 29, 1919, and its bearing upon the Einstein theory of gravitation" (illustrated lecture) by Dr. L. A. BAUER;

(5) Informal social and round table meeting at the Ohio Union following the evening lecture. Topic: "Freshman mathematics to meet the changing high school mathematics as presented for entrance to college." Discussion led by Mr. H. M. BEATTY, Ohio State University.

Abstracts of the papers and discussions follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper, Professor Borger presents two constructions: (*a*) The Peano construction for curves in rectangular coördinates; (*b*) A geometric method for the construction of all curves of the form $\rho = P(f_1, f_2, f_3, f_4, f_5, f_6)$, P being a polynomial and $f_1, \cdots f_6$ the six trigonometric functions. The construction may be applied to curves in parameter form and thus a class of curves not comprehended in (*a*) may be included. Thus most of the curves ordinarily arising in an analytic geometry course may be plotted without the cumbrous processes of the method by means of a table. For curves in polar coördinates additional power in determining their properties is gained.

2. Professor Lunn spoke of the blending of geometry, mechanics, and geometric optics which forms the experimental basis of elementary physical theory, emphasizing the meaning of equality of lengths in different places, of time intervals, and of the Pythagorean proposition as an experimental result, illustrating by contrast with the non-Euclidean character of geodesy on a large scale. He then showed how similar considerations led Einstein to consider space and time measurements as aspects of a single geometry, whose approximately Euclidean nature adapts it to the description of a wide range of physical relations and where the departures from the Pythagorean relation make it possible to define mass and motion and deduce the laws of hydrodynamics as geometric theorems.

3*a*. A brief historical survey of the problem of the relation between matter and ether was presented by Professor Blake including the Michelson-Morley experiment and the recent experiments of Majorana. The Fitzgerald-Lorentz contraction was then discussed and the way in which the restricted principle of relativity of Einstein (1905) accounted for all the negative results of experimenters was given. Brief comment was then made upon the generalized principle of relativity and it was shown that, if a generalized principle of least action for a four dimension continuum is to include all the laws of physics, one and the same Hamiltonian function must explain them, including the laws of radiation rather than breaking up that function into a series of additive independent terms.

3*b*. President Chamberlain discussed a new type of instrument for use in experiments on the ether drift. A combination of interferometer and diffraction grating produces a new type of interference possessing a high sensitiveness and freedom from disturbances. An interference system shown in the diffracted image produced by a grating, the lines of which are parallel with the interference fringes, consists of a narrow band of light crossed by sharp interference bands. The sensitiveness of the instrument depends upon the distance between the interference system and the diffraction grating. As the interfering paths are closely parallel over the greater part of their length the instrument is quite free from disturbance.

3*c*. Professor Miller reviewed the history of the experiments relating to ether drift which have been conducted at Cleveland, beginning with the famous Michelson-Morley experiment, down to the later Miller-Morley experiment.

Lantern slides were used to illustrate the apparatus and methods. Incidents of historical interest were related and particular attention was called to a persistent occurrence at the last experiments of a small displacement of the fringes, far less than the theory calls for, which has never been satisfactorily explained.

4. This lecture was fully reported in *Science*, March 26, 1920, pp. 301–311.

5. In the round table discussion, Mr. Beatty summarized the replies to questionnaires mailed out to about fifty of the leading high schools of Ohio. These reveal a tendency to minimize the amount of mathematics required for graduation. One unit each of algebra and geometry is required. In most cases one half unit each of advanced algebra and soild gometry is offered as an elective, but in many cases is not elected by the pupil. There is a tendency for more pupils to enter college deficient in a half-unit or more of mathematics. There was a feeling expressed that the same care in selecting teachers of mathematics was not exercised nor the same respect accorded mathematics as was done in former years. The opinion seemed to prevail that there was no more reason for discouragement over results in mathematics than in other subjects.

G. N. ARMSTRONG, *Secretary-Treasurer.*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION.

The fourth regular meeting of the Rocky Mountain Section was held at Colorado College, Colorado Springs, Colorado, April 2, 3. There were two sessions, presided over by Professor C. H. Sisam.

The attendance was twenty-five, including the following fourteen members of the Association: I. M. DeLong, J. C. Fitterer, W. H. Hill, H. A. Howe, Claribel Kendall, G. H. Light, J. Q. McNatt, S. L. Macdonald, O. A. Randolph, H. E. Russell, C. H. Sisam, C. S. Sperry, C. E. Stromquist, J. W. Woodrow.

The officers appointed for the meeting to be held at Denver in 1921 are: Chairman, H. A. HOWE, Denver University; Vice-chairman, W. H. HILL, Greeley High School; Secretary-Treasurer, G. H. LIGHT, Univ. of Colorado.

The following eight papers were read:

(1) "Some physical correlations in a group of one hundred S. A. T. C. men" by Professor J. C. FITTERER;

(2) "Families of curves whose evolutes are similar curves" by Professor G. H. LIGHT;

(3) "Grades for different placings of ears of corn" by Professor W. V. LOVITT;

(4) "Ionization in the mercury arc" by Professor J. W. WOODROW;

(5) "Discussion of the cycloidal curve" by Mr. J. Q. MCNATT;

(6) "Projective differential geometry in a four space" by Professor W. V. LOVITT;

(7) "The teaching of logarithms and slide rule in the first year of high school" by Professor C. E. STROMQUIST;

(8) "On ruled surfaces whose asymptotic curves are cubics" by Professor C. H. SISAM.

1. Professor Fitterer presented correlation tables which were computed between stature and stride, stride and weight, weight and stature. The correlation coefficient in the first was 0.29, in the second very nearly zero, and in the third 0.55. The average age was 19.3 years, average weight was 136 pounds, average stature was 5′ 8″, average stride was 5.7 ft. A hypsobaric coefficient (weight in pounds per foot of stature) was also found, which averaged 24 pounds per foot.

2. Professor Light's paper appears elsewhere in this issue.

3. Numerical grades were given by Professor Lovitt for different placings of any number of ears. The results are determinate, though arbitrary. The results are in use and are giving satisfaction with competent corn judges.

4. It was assumed by Professor Woodrow that (*a*) An electron, on the average, loses all of its translatory energy at each impact; (*b*) The molecules are capable of storing up energy, *i.e.*, after the energy is received, it is radiated by electromagnetic waves at a rate which is proportional to the instantaneous energy, and additional increments of energy can be added by successive impacts of different electrons with the same molecule; (*c*) The molecule can also receive energy which has been radiated from the surrounding molecules and which is proportional to the fourth power of the temperature of the gas or vapor. From these assumptions, the following equation was obtained

$$X = \frac{H}{I} p^{1/3}(K - p^{2/3}).$$

Where X is the electric force, I is the current, p is the pressure of the gas, and H and K are constants.

5. Mr. McNatt gave methods of constructing the cycloid and its evolute.

6. Given the linear differential equation of order five

$$y^{(5)} + 5p_1y^{(4)} + 10p_2y^{(3)} + 10p_3y^{(2)} + 5p_4y^{(1)} + p_5y = 0,$$

Professor Lovitt found invariants and covariants for the transformation $y = \lambda(x)\bar{y}$, $\xi = \xi(x)$. Some geometric interpretations were given.

7. Professor Stromquist suggested the following course for the first year of high school: (*a*) Tabulation and graphing of functions; (*b*) Meaning of positive and negative exponents, applying the four rules; (*c*) Square root of arithmetical numbers; (*d*) Logarithms, based on exponents; (*e*) Slide rule. The course has been successful in the Laramie High School.

8. Professor Sisam classified completely, and discussed the properties of, the ruled surfaces whose asymptotic curves are gauche cubics.

G. H. LIGHT, *Secretary-Treasurer.*

MATHEMATICS AND LIFE INSURANCE.[1]

By PERCY C. H. PAPPS,[2] Newark, N. J.

For an actuary to prepare an address for an association of mathematical teachers is in many respects more difficult than to prepare a paper for a society of actuaries. The thought has occurred to me that it might be interesting if I endeavor to show to what extent mathematics is used in actuarial work. I know that there is a very general idea that an actuary must be a skilled mathematician. In fact, it may not be too much to say that in many instances it is taken for granted that a skilled mathematician must of necessity make an expert actuary. Except for unusual computations, an elementary knowledge of mathematics is all that is required to understand the actuarial formulæ used in daily work. Some of the simpler formulæ and the manner in which they are derived will first be described and I will then outline some of the applications of the higher mathematics to life insurance work.

One of the most common fallacies which we meet is that the expectation of life is used in actuarial calculations. So far as I am aware, the only use which is ever made of the expectation of life is in comparing the effects of mortality under different mortality tables. It is rather generally supposed that in order to arrive at the single premium for an insurance of $1,000, payable upon the death of a man age 35, we take the expectation of life at age 35 and ascertain what sum of money now invested will at the end of the term of years equal $1,000 at the assumed rate of interest. A single premium thus computed is smaller than the true single premium. This can be proved mathematically, but it can be readily proved by general reasoning. In the first place the expectation of life of a man age 35 is the mathematical mean of the number of years lived by a large number of men now of age 35. According to the "American Experience Table of Mortality," the expectation of life at age 35 is about thirty-two years, and it may be supposed that in the computations there is one life that lived twenty-two years, and one which lived forty-two years, the mean of these two lives being thirty-two years. At 3% interest $521.89 is the present value of $1,000 due at the end of twenty-two years, and $288.96 is the present value of $1,000 due at the end of forty-two years. $810.85 is therefore the amount that would be required to be placed at interest at 3% in order to pay the two claims of $1,000 each at the end of twenty-two and forty-two years respectively. If in place of using the actual assumed duration for these two lives we used the expectation of life, which is thirty-two years, we would find that $388.34 is the present value of $1,000 at the end of thirty-two years, or $776.68 is the present value of $2,000 payable at the end of thirty-two years. The present value of these two cases when the expectation of

[1] Read before the Association of Mathematics Teachers of New Jersey, November 20, 1915. Two sections enclosed between asterisks * were added in proof sheets.

[2] Mathematician for the Mutual Benefit Life Insurance Company.

life is used is therefore $34.17 less than the correct amount. As a matter of fact, the expectation of life could only be correctly used if we assumed interest at the rate of 0%. When we assume any interest we are dealing with a geometrical rather than an arithmetical progression, and as stated before, the expectation of life is an arithmetical mean.

To find the single premium for $1,000 of insurance on a life now age 35, we require to know the rate of interest which it is safe to assume will be earned by the company for many years to come, and it is also necessary that we should have a table showing the rates of mortality likely to be experienced in the future, or, to put it perhaps more correctly, a table of mortality showing rates which will be not less than those likely to be experienced in the future. Before proceeding further it may be of interest briefly to outline the method employed in compiling mortality tables.

In the first place, by ascertaining for each age of life how many lives, how many policies, or how much insurance, as the case may be, was exposed to the risk of death, and how many lives, policies, or how much insurance was actually cancelled by death, we can ascertain for each age the rate of mortality. Even with a large amount of data, these mortality rates will exhibit accidental fluctuations, so that the data as first ascertained must be suitably graduated. It is not sufficient to ascertain the rate of mortality at each age, for it is necessary that we should have a table to show us out of a certain number living at one age how many will be living at some future age. This is done by taking an arbitrary number, such as 100,000 lives, at age ten, let us say, and by the rate of mortality at age ten we find out the number dying between ages ten and eleven. Subtracting this from 100,000, we have the number living at age eleven. Proceeding in this manner, we have a table showing the number living and the number dying at each age. We are then in a position to proceed to the calculation of single premiums and annuities. It may be mentioned in passing that the rate of mortality is not the only function or the usual function used in the graduation of a mortality table. It would be possible, for example, to start with a fixed radix and use the ungraduated rates to find the number living at each age, and then graduate the figures thus ascertained.

To find the single premium for an insurance of unity on a life now of age 35, we have theoretically to ascertain the cost of providing insurance between ages 35 and 36, between ages 36 and 37, and so on, until we come to the last age shown in the mortality table. In other words, we have a large number of computations to make, the sum of which will give us the single premium required. From our life table we know the number dying between ages 35 and 36. It is assumed that the claims will be paid at the end of the year. If we discount the amount required to the beginning of the year and divide this by the number of lives shown by the mortality table to be living at age 35, we would have the cost of insurance on a single life between ages 35 and 36. We can ascertain the number dying between ages 36 and 37, discount for two years the amount required to pay these claims, divide by the number living at age 35, and we have the cost of providing

the insurance between ages 36 and 37. This is the way in which the computations are made in theory. As a matter of fact, by means of what are known as commutation tables the calculations are made very expeditiously.

The commutation tables consist of several columns. If we take the number dying between ages 35 and 36 and multiply this by the present value of a unit payable thirty-six years hence, we will find the value of the C column for age 35. If we multiply the number living at age 35 by the present value of a unit due at the end of thirty-five years, we will have the value of the D column for age 35. If we then divide the value of the C column by the value of the D column for age 35, we will have the probability of a life age 35 dying between ages 35 and 36, discounted for one year. We may start from the bottom of the C column and form what is known as the M column by continuous addition of the values in the C column. Opposite age 35 in this column we will then have the sum of the values for the C column from age 35 to the end of the table. If this value in the M column is divided by the value for age 35 in the D column, we will immediately have the single premium for an insurance of unity at age 35. A similar summation of the D column gives what is known as the N column, and dividing any value in the N column by the corresponding value in the D column gives the value of a life annuity for the age selected. If, for example, we wished to obtain the value of an annuity running for ten years on a life age 35, we could take the sum of the values in the D column for ages 35 to 44 inclusive, and divide this by the value in the D column at age 35. It is a simpler matter, however, to subtract from the value in the N column at age 35 the value in the N column at age 45, which will immediately give us the sum of the ten values in the D column.

It has been explained that the single premium for an insurance at age 35 is merely the sum of the several costs of providing insurance payable in the event of a life dying in every year of life, and it is of course impossible that the event can happen more than once. In the same way, the value of an annuity of unity a year is the sum of the present values of the probabilities of a life living to each age to the end of the mortality table.

There are comparatively few who can afford to buy their insurance by means of a single cash payment, and a large proportion of the policies sold are what are known as Ordinary Life policies, where the premiums are payable during the continuance of the contract. Knowing the single premiums and life annuities, it is a simple matter to ascertain the annual premium required. It may be supposed that a company sells $1,000 of insurance and that the policyholder sells to the company a life annuity of a certain amount. The present value of the one must be equal to the present value of the other. If, therefore, we divide the single premium by the value of a life annuity of unity, the quotient will be the annual premium required. Under a Twenty Payment Life policy the premiums are limited to twenty years, so that the annual premium is ascertained by dividing the single premium by the present value of an annuity of unity running for twenty years. It may be mentioned that under an ordinary annuity the payments to the annuitant are made at the end of each year; under a life insurance

contract the premiums are paid at the beginning of each year. Care must be taken, therefore, that in calculating the value of annual premiums we use an annuity providing for the payments at the beginning of each year. On account of this difference, there are two forms of the N column of the commutation tables, which are likely to confuse the student until the difference is recognized.

The calculation of single and annual premiums for endowment insurances offers no difficulties. It must be remembered that a Twenty-Year Endowment insurance is made up of two parts. We have Twenty-Year Term insurance and what is known as a Twenty-Year Pure Endowment. Under the term insurance the claim is paid if the life dies during the twenty years, otherwise the contract becomes valueless at the end of twenty years. Under the Pure Endowment the contract is valueless in the event of the death of the insured, but is paid in full if the insured survives the twenty years. The single premium for the term insurance is ascertained for age 35 by taking the sum of twenty values in the C column from age 35 to 54 inclusive, and dividing by the value in the D column for age 35. The sum of the values in the C column is of course ascertained by subtracting the value in the M column at age 55 from the value in the same column for age 35. The single premium for the pure endowment at age 35 is ascertained by dividing the value in the D column for age 55 by the value in the same column for age 35.

We now come to the question of reserve. We have in the business of life insurance a very good illustration of the old saying, that a little knowledge is a dangerous thing. If the functions of the reserve on a life insurance policy had been well understood, this country would have been saved the enormous dissatisfaction and hardships caused by the failure of assessment insurance and fraternal orders. These concerns operated on what is known as the "reserve in the pocket" plan. That is, the members were supposed to keep the reserve in their own pockets. This will be referred to later. The reserve on a life insurance policy may be explained in several different ways. It has already been shown that at the inception of a contract of life insurance on the Ordinary Life plan, the single premium is exactly equal to the present value of the annual premiums. Consider the conditions of affairs when the policy has been in force one year. The value of the insurance granted by the company is greater than it was at the inception of the contract, for the insured is one year older, and consequently one year nearer his death. On the other hand, the value of the future premiums which the insured has agreed to pay to the company has decreased for the reason that the insured is one year older and certainly has one less premium to pay. The company's liability represented by the present value of the insurance granted is therefore in excess of the company's asset, which is represented by the present value of the annual premiums yet to be received. Therefore, in order that the company may not be insolvent, it must have on hand a sum equal to the difference between the value of the insurance granted and the value of the premiums yet to be received. This is technically known as the reserve.

The question may arise as to where the reserve comes from. It is readily seen that as a man grows older the yearly cost of providing insurance, provided a

new contract is taken each year, must steadily increase. Under an Ordinary Life policy it does not increase for the reason that the cost is equalized. The first premiums are more than sufficient to pay for carrying the insurance, and it is the balance that goes into the reserve. Furthermore, in the event of death the company is called upon to provide also the difference between the face of the contract and the reserve. As the life grows older, the reserve steadily increases, so that the amount at risk steadily decreases. It thus comes about that in spite of the increasing age of the insured, the annual premium plus the interest on the reserve is always sufficient to pay for the yearly cost of insurance and provide for the necessary increase in the reserve. At age 96, which is the limit of life according to the American Table of Mortality, the entire reserve exactly equals the face of the policy.

The assessment plan of insurance is correct in theory, for there is no theoretical reason why a policyholder should not pay each year just the mere cost of providing his insurance. The objections are practical ones. The net premium at 3 per cent. interest for one year's insurance of $1,000 at age 35 is $8.68. At age 75 it is $91.62. If a man were so fortunate, or unfortunate, as to live ten years more, he would have to pay for one year's insurance $228.69. Many assessment societies have attempted to provide insurance at level rates at an amount more than sufficient to cover the cost at the young ages, but without holding any reserve which would enable them to continue that cost at the older ages. They have claimed that they left the reserve in their members' pockets. If that was the case it was permissible for them to allow the members to withdraw and retain the reserve in their pockets. The great mistake which they made was that when members died they forgot all about the reserve being in the members' pockets, and paid the full amount of the certificates to the beneficiaries. It is quite possible, if not probable, that many fraternal societies which have been wound up on account of insolvency, would be today in existence if they had paid the beneficiaries merely the difference between the face of the certificates and the amount which the deceased members had retained as reserve in their pockets.

From what has been said you will readily see that a knowledge of elementary algebra up to and including the progressions is sufficient to enable anyone to grasp the ordinary formulæ necessary for calculating premiums and reserves. We will now pass on to consider a few cases where some knowledge of the higher mathematics is necessary. From what has been said it will readily be seen that the calculation of a single premium, for example, where we have merely the mortality table without any commutation columns would be an exceedingly laborious undertaking. Let us suppose that we wish to calculate the cost of an annuity on the life of a woman after the death of her husband. Now in the first place, we know that the mortality on the life of an annuitant is very different from the mortality on the life of one who is insured. It is necessary, therefore, to suppose that our prospective annuitant will experience a mortality according to an annuity table, while her husband will be one whose mortality we will have to measure by a table on which our premium rates for life insurance are

based. We are therefore forced to consider a computation which will involve two mortality tables as well as a rate of interest. In such a case the integral calculus is of inestimable benefit. By means of the calculus we can use a formula of approximate summation which will give results surprisingly close to an exact calculation. The use of one of these formulæ can be best described by taking a simple example of a single premium for life insurance on a single life. It has already been shown how by exact calculation we can ascertain the present value of a life age 35 dying in the first, second, third years, etc., from the present time, and add all these values together. By means of the approximate summation formulæ we can ascertain the present value of the probability of a life dying in the eighth year, the twenty-fourth, fortieth, forty-eighth and fifty-sixth years; multiplying each of these values by proper coefficients and taking eight times the sum of the results will give us a practically correct result. It is true that the approximate summation formulæ are based on the integral calculus, and to understand the derivation of the formulæ a knowledge of the differential and integral calculus is required; yet, as a matter of fact, these approximate summation formulæ are in daily use by those who have no knowledge of either the differential or integral calculus, but who have been trained to apply the formulæ.

* Starting with the well known Maclaurin's Theorem

$$u_x = u_0 + x\frac{du_0}{dx} + \frac{x^2}{\lfloor 2}\frac{d^2u_0}{dx^2} + \frac{x^3}{\lfloor 3}\cdot\frac{d^3u_0}{dx^3} + \text{etc.}$$

and integrating we get

$$\int u_x dx = xu_0 + \frac{x^2}{\lfloor 2}\frac{du_0}{dx} + \frac{x^3}{\lfloor 3}\frac{d^2u_0}{dx^2} + \frac{x^4}{\lfloor 4}\frac{d^3u_0}{dx^3} + \text{etc.}$$

Taking the integral between the limits $-n$ and $+n$

$$\int_{-n}^{n} u_x dx = 2nu_0 + \frac{2n^3}{\lfloor 3}\frac{d^2u_0}{dx^2} + \text{etc.} \qquad (A)$$

All but the first two terms on the right hand side of the above equation may be ignored and it may be assumed that

$$\int_{-n}^{n} u_x dx = au_0 + b(u_{-n} + u_n). \qquad (B)$$

Expanding by Maclaurin's Theorem this becomes

$$\int_{-n}^{n} u_x dx = au_0 + b\left(2u_0 + \frac{2n^2}{\lfloor 2}\frac{d^2u_0}{dx^2} + \text{etc.}\right). \qquad (C)$$

Equating the coefficients of expressions (A) and (C) it will be seen that $a+2b=2n$ and $bn^2 = n^3/3$. From this it follows that $a = 4n/3$ and $b = n/3$. Substituting these values in (B) we get

$$\int_{-n}^{n} u_x dx = \frac{n}{3}(u_{-n} + 4u_0 + u_n)$$

or

$$\int_0^{2n} u_x dx = \frac{n}{3}(u_0 + 4u_n + u_{2n}).$$

From this it follows that

$$\int_0^{\infty} u_x dx = \int_0^{2n} u_x dx + \int_{2n}^{4n} u_x dx + \int_{4n}^{6n} u_x dx + \text{etc.}$$

$$= \frac{n}{3}\{(u_0 + 4u_n + u_{2n}) + (u_{2n} + 4u_{3n} + u_{4n}) + \text{etc.}\}$$

$$= \frac{n}{3}\{u_0 + 2(u_{2n} + u_{4n} + u_{6n} + \text{etc.}) + 4(u_n + u_{3n} + u_{5n} + \text{etc.})\}.$$

In the example illustrating this formula given in the *Institute of Actuaries' Text Book*, n has been given the value 15 in ascertaining the value of a continuous life annuity at age 30. Instead of computing the probabilities of living every duration from one to sixty-five years only four values are used, namely, $v^{15}l_{45}/l_{30}$, $v^{30}l_{60}/l_{30}$, $v^{45}l_{75}/l_{30}$ and $v^{60}l_{90}/l_{30}$ where l_{30}, for example, represents the number living at age 30 according to the mortality table used in the computations and v^{15} the present value of a unit payable fifteen years hence. The value of the annuity obtained by using the formula is 20.3899 while the true value is 20.3919; an error of only .002.

By taking more terms in formulæ (A) and (B), other summation formulæ may be derived. The one referred to above is thus derived. The formula is

$$\int_0^{\infty} u_x dx = n(.28u_0 + 1.62u_n + 2.2u_{3n} + 1.62u_{5n} + .56u_{6n} + 1.62u_{7n}).$$

In applying this formula n is so taken that $7n$ falls just within or just beyond the limit of life shown by the mortality table.*

In the graduation of mortality tables many methods are employed. We may take the sum of several values, assume that the average of these values is applicable to the central age of the group, plot the points, and by drawing a smooth curve through the points a series of graduated values may be obtained. The graduation of a set of values may also be accomplished by means of a summation method. For example, if we take three times the sum of three consecutive values from ten times the central value, the result will be a very rough graduation. By taking these values and summing each five adjacent values, repeating this operation three times and taking eight one thousandths of the final result a smoothly graduated series of values will result, unless the ungraduated values are very uneven. An extension of this principle enables us to derive formulæ where the graduated value at any age will be made up of suitable portions of the values extending as far as fourteen terms on each side of the central value. A graduation of a mortality table may also be effected by assuming that a certain mathematical law will hold. One well-known law involves four arbitrary constants, which must

be determined from the ungraduated data. The age, of course, is the variable, and by choosing suitable constants it has been found possible to graduate many mortality tables, so that the number living at any age is determined by the mathematical law. One great advantage which follows the use of one of these mathematical laws is that where several lives are involved in the computations it is possible to substitute for the lives of different ages the same number of lives of equal ages, the corresponding age being ascertained from the graduation constants. In the selection of the graduation constants so that the graduated values may fit as closely as may be to the ungraduated data, application has been made of the theory of moments, the method of least squares and frequency curves.

*If u_0 be a central value and u_{-1} and u_{+1} be adjoining values then $10u_0 - 3(u_{-1} + u_0 + u_{+1})$ is the simple formula mentioned above, which reduces to $\acute{u}_0 = -3u_{-1} + 7u_0 - 3u_{+1}$, where $\acute{u}_0$ is a roughly graduated value of u_0. A further development of this principle gives Woolhouse's formula, which is as follows:

$$\acute{u}_0 = (1/5)\{u_0 + .96(u_{-1} + u_{+1}) + .84(u_{-2} + u_{+2}) + .28(u_{-3} + u_{+3}) + .12(u_{-4} + u_{+4}) - .08(u_{-5} + u_{+6}) - .12(u_{-7} + u_{+7})\}.$$

The lengthy formula referred to above was described by the author in a paper published in the December 1918 number of the *Publications of the American Statistical Association*. This formula is as follows:

$$\begin{aligned}\acute{u}_0 = (1/5)\{&u_0 + .9344(u_{-1} + u_{+1}) + .7264(u_{-2} + u_{+2}) + .4384(u_{-3} + u_{+3}) \\ &+ .1744(u_{-4} + u_{+4}) - .0976(u_{-6} + u_{+6}) - .1136(u_{-7} + u_{+7}) \\ &- .0736(u_{-8} + u_{+8}) - .0256(u_{-9} + u_{+9}) + .0128(u_{-11} + u_{+11}) \\ &+ .0144(u_{-12} + u_{+12}) + .0080(u_{-13} + u_{+13}) + .0016(u_{-14} + u_{+14})\}.\end{aligned}$$

It will be found that the sum of the coefficients within the bracket is equal to 5.

The simplest form of the mathematical law of mortality is that of Gompertz which, expressed logarithmatically, is $\log l_x = \log k + c^x \log g$ where l_x represents the number living at age x and k, c and g are constants.

A first modification of this law derived by Makeham is

$$\log l_x = \log k + x \log s + c^x \log g.$$

A second modification is

$$\log l_x = \log k + x \log s + x^2 \log \omega + c^x \log g.*$$

It will be recognized that I have touched but briefly on many subjects which can only properly be covered in an extensive actuarial library. It should not be forgotten, however, that very few actuaries are ever called upon to graduate a mortality table; consequently it is of more importance to an insurance company that the actuary should be a man of good business judgment, with some original-

ity, able to meet the public, and compose a decent letter. He must have the mathematical ability to enable him to pass his actuarial examinations and qualify as an actuary, but so far as his practical work is concerned, in the training of his actuarial department for the regular routine work of the office he can forget if he will practically all of his higher mathematics. When an actuary is called upon, as is frequently the case, to take his part in the executive work of the company, it is necessary that he should have some knowledge of life insurance law, the principles of accounting, and a general knowledge of investments and finance. From what I have said I believe that you will agree with me when I say that an actuary is not necessarily an expert mathematician, and that an expert mathematician will not necessarily make a good actuary.

NOTE ON THE ROOTS OF THE DERIVATIVE OF A POLYNOMIAL.

By W. H. ECHOLS, University of Virginia.

(Read before the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America, May 15, 1920.)

The problem of determining regions to which are confined the roots of the derivatives of functions has received a great deal of attention. Lucas established the fact[1], by a mechanical proof, that the roots of the derivative of any polynomial are confined to the smallest convex region enclosing the roots of the polynomial. Maxime Bôcher gave a geometrical demonstration of this.[2] He noted that the roots of the derivative of any cubic equation are the foci of the ellipse tangent at their mid-points to the sides of the triangle of the roots of the cubic; and remarked that it might be possible to associate the roots of the derivative of any polynomial with the foci of a higher plane curve. B. Z. Linfield gave a beautiful demonstration, in a paper read at the meeting, in St. Louis last December, of the American Mathematical Society (*Bulletin*, p. 264), that the roots of the derivative of any polynomial of degree n were the foci of a curve of class $n - 1$ touching at their mid-points the segments joining, two and two, the roots of the polynomial. Recently J. L. W. V. Jensen stated,[3] without demonstration,

[1] F. Lucas, "Géométrie des polynômes," *Journal de l'Ecole Polytechnique*, 1879, cahier 46, tome 28, pp. 1–33.—EDITOR.

[2] M. Bôcher, "Some propositions concerning the geometric representation of imaginaries," *Annals of Mathematics*, March, 1893, vol. 7, pp. 70–72.—EDITOR.

[3] In a letter to the writer Dr. J. L. Walsh, of Harvard, says: "Jensen originally stated his theorem without proof, in *Acta Mathematica*, vol. 36 (1912), p. 190. Apparently the theorem remained unnoticed until Professor D. R. Curtiss called attention to it in a paper presented at a meeting of the American Mathematical Society; his abstract was published in the current volume of the *Bulletin*, pp. 61–62. Professor Curtiss presented further results in April (see the June *Bulletin*, p. 392). I also have some further results which I presented to the Society in December; the abstract of which is in the March *Bulletin*, p. 259. My paper has been accepted for publication in the *Annals of Mathematics*; it contains a proof of Jensen's theorem based on mechanical considerations. Your own proof is surely different from mine (in form but not in substance). I do not know whether it is different from Curtiss's. I was not aware an algebraic proof could be given so simply. No proof of Jensen's theorem has yet been published."

that the complex roots of the derivative of a polynomial with real coefficients lie inside circles having the segments joining the pairs of conjugate roots of the polynomial as diameters. It is the purpose of this note to present a complete demonstration establishing the truth of this statement.

THEOREM. If $f(z)$ is a polynomial whose coefficients are real numbers, then the imaginary roots of the derivative $f'(z)$ lie on or within circles whose diameters are the segments joining the pairs of conjugate imaginary roots of $f(z)$.

Proof. Let

$$f(z) \equiv L_1^{p_1} L_2^{p_2} \cdots L_m^{p_m} \cdot Q_1^{q_1} Q_2^{q_2} \cdots Q_n^{q_n},$$

wherein p_r and q_r are positive numbers, and

$$L_r \equiv z - a_r, \qquad Q_r \equiv (z - b_r)^2 + c_r^2,$$

a_r, b_r, c_r real numbers. Taking the logarithm and differentiating, the derivative is

$$f'(z) = f(z) \cdot \Sigma \left(\frac{p_r}{z - a_r} + 2q_r \frac{z - b_r}{Q_r} \right).$$

Realizing the denominators ($z \equiv x + iy$)

$$\frac{p_r}{z - a_r} = \frac{p_r}{x - a_r + iy} = p_r \frac{(x - a_r) - iy}{(x - a_r)^2 + y^2},$$

and

$$\frac{z - b_r}{Q_r} = (x - b_r) \frac{(x - b_r)^2 + y^2 + c_r^2}{D_r} - iy \frac{(x - b_r)^2 + y^2 - c_r^2}{D_r},$$

where, for brevity,

$$D_r \equiv \{(x - b_r)^2 - y^2 + c_r^2\}^2 + 4(x - b_r)^2 y^2.$$

Hence, equating to zero the real and imaginary components of the sigma factor of $f'(z)$, we have the necessary and sufficient conditions for those roots of $f'(z)$ which are not common to $f(z)$,

$$\Sigma \left\{ p_r \frac{x - a_r}{(x - a_r)^2 + y^2} + 2q_r (x - b_r) \frac{(x - b_r)^2 + y^2 + c_r^2}{D_r} \right\} = 0,$$

$$y\Sigma \left\{ \frac{p_r}{(x - a_r)^2 + y^2} + 2q_r \frac{(x - b_r)^2 + y^2 - c_r^2}{D_r} \right\} = 0.$$

The real roots are furnished by $y = 0$ and the first condition. For $y \neq 0$ it is impossible for the second condition to vanish unless the point x, y is inside one of the circles

$$(x - b_r)^2 + y^2 = c_r^2.$$

The multiple roots of the polynomial are of course also roots of the derivative and these are on the circles. This justifies the statement.

May 1, 1920.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

34. Given the mixed integral and functional equation
[*1917*, 134, 341; 1920, 114]

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

I. REPLY BY ELIJAH SWIFT, University of Vermont.

Assume that $f(x)$ is developable in a power series in the neighborhood of the origin

$$f(x) = A_0 + A_1x + \cdots + A_nx^n + \cdots.$$

Substituting this value for $f(x)$ and integrating we have

$$A_0h + A_1\frac{h^2}{2} + \cdots + A_n\frac{h^{n+1}}{n+1} + \cdots = A_0h + A_1\frac{h^2}{2} + \cdots + \frac{h}{6}A_n\left[4\cdot\frac{h^n}{2^n} + h^n\right] + \cdots;$$

or transposing,

$$\sum_{n=2}^{\infty} A_nh^{n+1}\left\{\frac{1}{n+1} - \frac{1}{6}\cdot\frac{4}{2^n} - \frac{1}{6}\right\} = 0.$$

If this is to hold for all values of h near 0, the coefficients of the terms of the series must all vanish. Obviously this is the case for $n = 2$ and $n = 3$. This leads to the equation

$$A_n\left(\frac{1}{n+1} - \frac{1}{6\cdot 2^{n-2}} - \frac{1}{6}\right) = 0.$$

Since

$$\frac{1}{n+1} - \frac{1}{6\cdot 2^{n-2}} - \frac{1}{6} < \frac{1}{n+1} - \frac{1}{6},$$

it is clear that for $n \geqq 5$, the coefficient of h^{n+1} cannot vanish unless $A_n = 0$. For $n = 4$, direct calculation gives the same result. Consequently *if the function $f(x)$ is analytic, it must be a polynomial of degree not larger than* 3.

II. REPLY BY A. A. BENNETT, Washington, D. C.

The problem as stated is ambiguous. The question depends upon whether h is to be some fixed value, as might be implied by the notation, or is a variable for all values of which the functional relation is to hold.

If h is constant, the solution obviously admits of an infinite number of independent parameters since only the definite integral and the value of the function at three points are involved. It is not difficult to select even an infinite number of values for which the above relation holds, while the function still retains an infinite number of independent parameters. In view of the application suggested, the following formulation of the problem may be of interest: To determine the character of $f(x)$ subject to the condition that the above relation holds for all values of h between 0 and H, (> 0), inclusive, and that $f(x)$ shall be defined in this interval as made up of a finite number of continuously joining analytic pieces—these pieces to be analytic even at the end points of the intervals in which they are used.

If h lies in the first interval (starting toward H from 0), it is at once verified that $f(x)$ is a polynomial of degree not greater than 3, but not otherwise restricted, so far as this interval is concerned. Expansion in series and comparison of undetermined coefficients suffices for this proof. Let a be the end point of this first interval. Let h be in the second interval, near a, so that $h/2$

is in the first interval. Suppose f_1, is the analytic continuation of $f(x)$ (defined in the first interval), which of course exists for the polynomial, and let f be the assumed function in the second interval. Now $f_1(x)$ will satisfy the condition of the problem as truly as $f(x)$. Subtracting, and denoting $f(x) - f_1(x)$ by $g(x)$, we have

$$\int_a^h g(x) = \frac{h}{6}[g(h)].$$

Expanding $g(h)$ in powers of $h - a$ and comparing coefficients to satisfy this condition, it is noted that $g(h)$ vanishes identically. Thus $f(x)$ coincides with $f_1(x)$ at the beginning of the second interval. In other words there is no break in the analytical representation. Thus a polynomial of at most the third degree but not otherwise specified is a solution and the only solution of the problem as restated.

III. Remarks by the Editor.

It is clear, as Professor Bennett states, that in case h is supposed to be a given constant, so little restriction is placed on the function $f(x)$ as to make the problem relatively uninteresting. When h is allowed to vary, however, it is still possible to interpret the problem in different ways, depending on the hypotheses to be made concerning $f(x)$. We may demand that $f(x)$ be analytic throughout a neighborhood of $x = 0$. This case has been considered thoroughly in both the preceding replies; all the algebraic steps are given in Professor Swift's reply. It follows, of course, that $f(x)$ must be a polynomial of degree $\leqq 3$.

We may demand that $f(x)$ consist at worst of a number of continuously joining analytic pieces. Professor Bennett shows that this extension produces no new solutions.

By his method it is possible to demonstrate also the following similar theorem: *If two functions which are continuous,* $0 \leqq x \leqq H$, *and satisfy the equation,* $0 \leqq h \leqq H$, *are identical in the interval* $0 \leqq x \leqq \epsilon$, *they are identical throughout the interval* $0 \leqq x \leqq H$.

To prove this, denote the difference of the two functions by $g(x)$, and let a be the greatest lower bound of the points for which $g(x) \neq 0$. Then for h slightly greater than a, we have, as in Professor Bennett's reply,

$$\int_a^h g(x)dx = \frac{h}{6}g(h).$$

From this equation it follows that $g(x)$ possesses a derivative, and that

$$hg'(h) - 5g(h) = 0;$$

hence

$$g(h) = Ah^5;$$

but as $g(a) = 0$, it follows that $g(h) = 0$. This contradicts the choice of the point a; no such point can exist and the theorem is proved.

In particular therefore, *if* $f(x)$ *is continuous,* $0 \leqq x \leqq H$, *and analytic at* $x = 0$, *it must be a polynomial of degree not greater than 3.*

It seems not unlikely that a similar result holds without the assumption of analyticity at any point. Two analogies make this suggestion worthy of consideration. In the first place, the equation which bears to the trapezoidal rule the same relation as the given equation to Simpson's rule:

$$\int_0^h f(x)dx = \frac{h}{2}[f(0) + f(h)]$$

is readily shown to have no solutions but linear functions, on the assumption of mere continuity. In the second place, if in the given equation both limits of integration, instead of the upper limit alone, be allowed to vary, the question may be answered without the use of analyticity. Let the equation

$$\int_a^\beta f(x)dx = \frac{\beta - \alpha}{6}\left[f(\alpha) + 4f\left(\frac{\alpha+\beta}{2}\right) + f(\beta)\right]$$

be satisfied for all values of α and β in an interval throughout which $f(x)$ possesses a fifth derivative. By repeated differentiation we readily obtain

$$f^{IV}(\beta) = \frac{\beta - \alpha}{6}\left[\frac{1}{8}f^{V}\left(\frac{\alpha+\beta}{2}\right) + f^{V}(\beta)\right] + \frac{1}{6}\left[\frac{5}{4}f^{IV}\left(\frac{\alpha+\beta}{2}\right) + 5f^{IV}(\beta)\right].$$

Letting $\beta = \alpha$, we have $f^{IV}(\alpha) = 0$. Since this must hold for an arbitrary α in the interval under consideration, $f(x)$ can be at worst a cubic polynomial.

In view of the previous remarks, it seems reasonable to adopt as a goal with reference to this question the proof that if the equation holds for some range of the variable h, the function $f(x)$ can be only a polynomial of degree $\leqq 3$, under restrictions as light as possible—*e.g.*, that $f(x)$ should be continuous and possess a stated number (as small as it can be made) of derivatives.

DISCUSSIONS.

In the study of a geometric locus, it is frequently useful to have an analytic form of representation which shall exhibit the properties of the locus as regards its size and shape, without reference to its position. Such a form of representation, in the case of a plane curve, is furnished by the *intrinsic equation* of the curve —the functional relation which subsists between the length of arc measured from a fixed point of the curve to a variable point, and the radius of curvature at the variable point. Professor Light makes use of the intrinsic equation to study curves whose evolutes are similar to themselves. He obtains only the cases already known—the logarithmic spiral, and the cycloidal curves, and shows that no other examples exist whose intrinsic equations are of the special type $AR^n + BS^m - C = 0$. It would be of interest to know whether any other curves whatever can have this property.

In this department for May appeared a derivation of the formulæ for the tangents of the half angles of a triangle from the law of sines, by Professor Baudin, In the present number Professor Bohannan gives a method of obtaining tangent, sine and cosine of the half-angles by a geometric proof based on the use of inscribed and escribed circles.

Professor Poor recommends the more extensive use of directed lines and the method of projection in connection with elementary analytic geometry. In fact the use of such notions in the class-room is perhaps more prevalent than their appearance in text-books might indicate. Even in text-books however they are to be found here and there. The theorems stated by Professor Poor are now generally used in trigonometry to prove the formulæ for sines and cosines of sums and differences of angles. In some texts on analytic geometry they form the only mode of approach to transformation of coördinates by rotation; and they are frequently used in just the fashion suggested by Professor Poor to obtain the normal form of the equation of a straight line.

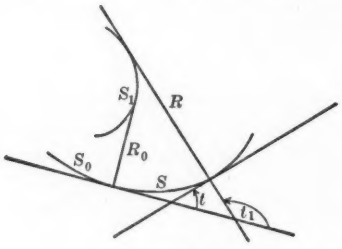
I. Note on Curves Whose Evolutes are Similar Curves.

By G. H. Light, University of Colorado.

It is known[1] that the evolutes of certain curves are similar curves. It is the purpose of this paper to give the general conditions that must be satisfied by

[1] Euler: "Investigatio curvarum quae evolutae sui similes producunt," *Comm. Petrop.* vol. 12, 1750, pp. 19–23.
Loria: *Spezielle algebraische und transcendente ebene Kurven*, Leipzig, 1902, p. 625.
Cesàro, *Vorlesungen über naturliche Geometrie*, Leipzig, 1901, pp. 36–37.

the base curve when this property holds true. Let the intrinsic equation of the base curve be $F(R, S) = 0$. Then, if subscript 1 applies to the evolute (see figure)



$$S_1 = R - R_0, \qquad t_1 = t + 90°.$$

Hence

$$R_1 = dS_1/dt_1 = dR/dt = R\,dR/dS.$$

Therefore, the equation of the evolute of $F(R, S) = 0$ is to be found by eliminating S and R from the three equations

$$F(R, S) = 0, \qquad R_1 = R\,dR/dS, \qquad S_1 = R - R_0. \tag{1}$$

If now the first of the three equations is solved for S, one obtains $S = S(R)$. Then, since

$$dR/dS = -F_S(R, S)/F_R(R, S)$$

the second equation of (1) states that

$$R_1 = -RF_S(R, S)/F_R(R, S).$$

Finally, use of the third of the equations of (1) gives

$$R_1 = -(S_1 + R_0)F_S(S_1 + R_0, S(S_1 + R_0))/F_R(S_1 + R_0, S(S_1 + R_0)).$$

which may be rewritten in the more symmetric form

$$R_1F_R + (S_1 + R_0)F_S = 0, \tag{2}$$

where it is, of course, understood that the arguments R and S in F_S and F_R have been replaced by $S_1 + R_0$ and $S(S_1 + R_0)$ respectively. Equation (2) is the form of the evolute most suitable for further use. Consider now the class of curves whose intrinsic equations are of the form

$$F(R, S) = AR^n + BS^m - C = 0, \tag{3}$$

where A, B and C are constants.

The relations between m and n and their exact values will be determined when their evolutes are similar curves.

Solving equation (3) for S, and remembering that $R = S_1 + R_0$, gives

$$S = \left(\frac{C - AR^n}{B}\right)^{1/m} = \left[\frac{C - A(S_1 + R_0)^n}{B}\right]^{1/m}.$$

Hence

$$F_S = mBS^{m-1} = mB\left[\frac{C - A(S_1 + R_0)^n}{B}\right]^{(m-1)/m}$$

and

$$F_R = nAR^{n-1} = nA(S_1 + R_0)^{n-1};$$

when the values thus found are substituted in the equation of the evolute, *i.e.* (2), the following equation results

$$nAR_1(S_1 + R_0)^{n-1} + mB(S_1 + R_0)\left[\frac{C - A(S_1 + R_0)^n}{B}\right]^{(m-1)/m} = 0.$$

If the factor $S_1 + R_0$ is discarded, one of the terms transferred to the right hand side of the equation and then both terms raised to the power $m/(m-1)$ the resulting equation of the evolute can be rewritten in the form:

$$\left(-\frac{An}{Bm}\right)^{m/(m-1)} R_1^{m/(m-1)}(S_1 + R_0)^{m(n-2)/(m-1)} + \frac{A}{B}(S_1 + R_0)^n = \frac{C}{B}. \qquad (4)$$

Hence, if the evolute and the base curves are to be similar,[1] *it must be possible to choose R_0 as zero.* This requires that the base curve have a cusp and that the origin be chosen at that point. When this is possible, a comparison of exponents of R and S in (3) with those of R_1 and S_1 in (4) gives the three following conditions on m and n:

$$n = m/(m-1),$$

$$m(n-2)/(m-1) = 0,$$

$$n = m.$$

The only solutions of these equations are $m = n = 0$ and $m = n = 2$. The first may obviously be discarded, since no intrinsic equation would exist. When $m = n = 2$, the resulting equation of the evolute is

$$AR_1^2 + BS_1^2 = CB/A.$$

In résumé[2]: *Given the family of curves*

$$AR^n + BS^m = C,$$

where $C \neq 0$, $m \neq 1$; when the equations of their evolutes are formed and the condition imposed that curve and evolute be similar, a first necessary condition is found to be $R_0 = 0$. The second is $m = n = 2$. This gives the cycloidal family.

As a second class of curves consider the family[3]

$$F(R, S) \equiv R^n - CS^m = 0. \qquad (5)$$

[1] The intrinsic equations of two similar curves differ only in the substitution of kR, kS for R, S, where k is a constant (Cesáro, l.c., p. 28).

[2] At various stages of the preceding proof it has been assumed that $B \neq 0$, $C \neq 0$, $m \neq 1$. The case $B = 0$ gives a circle, which is easily seen not to satisfy the conditions. The case $C = 0$ will be considered more at length. The case $m = 1$ may be carried through by the methods indicated, it will be found that necessarily also $n = 1$; the resulting equation $AR + BS - C = 0$ is a form of the equation of the logarithmic spiral.—EDITOR.

[3] This corresponds to the case $C = 0$ in (3), since the assumption that A also vanishes would lead to no real curve.—EDITOR.

Use of the defining equations of the evolute gives

$$F_R = nR^{n-1} = n(S_1 + R_0)^{n-1},$$

$$F_S = -mCS^{m-1} = -mC^{1/m}(S_1 + R_0)^{n(m-1)/m}.$$

From these values of F_S and F_R the equation of the evolute is found, from (2), to be

$$nR_1(S_1 + R_0)^{n-1} - mC^{1/m}(S_1 + R_0)(S_1 + R_0)^{n(m-1)/m} = 0.$$

An easy simplification of this equation reduces it to

$$R_1{}^m = C\left(\frac{m}{n}\right)^m (S_1 + R_0)^{2m-n}. \tag{6}$$

If this last equation is to represent a curve similar to the base curve (5), then it is again necessary—first, that $R_0 = 0$, *i.e.*, that a cusp exist on the base curve; and second, that the exponents correspond. This gives $m = n$. Hence[1] the result is obtained;

All curves whose intrinsic equations can be written $R^n = CS^m$ *have evolutes similar to themselves only when* $R_0 = 0$ *and when* $m = n$. *This gives the logarithmic spiral* $R^m = CS^m$.

II. Functions of Half-Angles of a Triangle.

By R. D. Bohannan, Ohio State University.

In the accompanying figure I is the center of the circle inscribed in the triangle ABC, and E is the center of the escribed circle touching the side BC and the prolongations of AB and AC. IH, IK, EF, EG are radii of these circles drawn to some of the points of tangency. If $BC = a$, $AC = b$, $AB = c$, $a + b + c = 2s$, then

$$AH = s - a; \qquad AG = s; \qquad CK = s - c; \qquad BK = s - b.$$

A circle on IE as diameter passes through C, B. If IK is prolonged to meet this circle in N, KN is equal to FE.

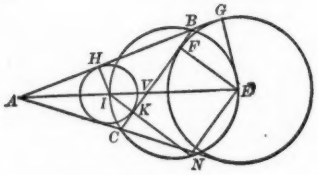
$$\tan\frac{A}{2} = \frac{IH}{AH} = \frac{EG}{AG}.$$

Hence

$$\tan^2\frac{A}{2} = \frac{IH}{AH}\cdot\frac{EG}{AG} = \frac{IH\cdot EG}{s(s-a)}$$

$$= \frac{IK\cdot FE}{s(s-a)} = \frac{IK\cdot KN}{s(s-a)}$$

$$= \frac{CK\cdot KB}{s\cdot(s-a)} = \frac{(s-b)(s-c)}{s(s-a)}.$$



[1] The cases excluded by the assumptions $m \neq 0$, $n \neq 0$, $C \neq 0$ in the course of the proof can be shown as before to yield nothing new.—Editor.

Let V be the intersection of AE and BC. Then

$$\sin^2\frac{A}{2}=\frac{IH}{AI}\cdot\frac{EG}{AE},\qquad \cos^2\frac{A}{2}=\frac{AH}{AI}\cdot\frac{AG}{AE}.$$

Since BI and CE (not drawn in the figure) are angle-bisectors,

$$\frac{AI}{IV}=\frac{AB}{BV},\qquad \frac{AE}{VE}=\frac{AC}{VC}.$$

But

$$IV\cdot VE=BV\cdot VC.$$

Hence

$$AI\cdot AE=AB\cdot AC=bc.$$

Therefore

$$\sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc},\qquad \cos^2\frac{A}{2}=\frac{s(s-a)}{bc}.$$

III. The Use of the Vector in Analytical Geometry.

By Vincent C. Poor, University of Michigan.

In a first course in analytical geometry it is necessary to adhere rather closely to the text, whatever the text. This is true not because of the nature of the subject matter but because of the mathematical immaturity of the students electing the subject. Important innovations thus furnish one excuse for another textbook.

In many of the textbooks on analytical geometry the directed line is not mentioned at all. This is deplorable from the point of view of the physicist, for the geometric interpretation of many physical quantities leads to simplicity in expression and clearness in comprehension. Aside from this need the subject of analytical geometry may, in my opinion, be much more easily and directly presented if a more extended use of the vector be made.

The ground work for this is to be found in some of our textbooks, *e.g.*, Woods and Bailey, *A Course in Mathematics*, Vol. I; Ziwet and Hopkins, *Analytic Geometry*. In their study of directed lines we find the equivalents of the following theorems:

Theorem I. *The projection of a line segment on another line is equal to the length of the line segment into the cosine of their included angle.*

Theorem II. *The projection of a broken line on another line is equal to the projection of the join of its end points.*

In a number of the books the fundamental theorem of the geometry of segments is deduced, namely

Theorem III. *Given three points, O, P, Q, on a directed line, then*

$$PQ=OQ-OP,$$

in magnitude and sense.

If O is taken as the origin and the coördinates x_1 and x_2 be assigned to the

points P and Q respectively we can write

$$PQ = x_2 - x_1$$

in magnitude and sense.

Although these three theorems are given in some of our textbooks, very little use is made of them in the further development of the subject. Students may well wonder why they have a place in the book at all. And yet if they were used the usual theorems of analytical geometry expressed in rectangular coördinates could be easily and rigorously proved, the proofs being general and not "piecemeal" as is the case in possibly all of our texts. A few illustrations will serve to show this.

After defining the rectangular coördinate system, we may consider a directed line, a vector, anywhere in the plane. Call its initial point $P(x_1, y_1)$ and its terminal point $P_2(x_2, y_2)$. Let d and θ be its length and vectorial angle respectively. Then the projections P_1' and P_2' of P_1 and P_2 on the axis OX will be given by the coördinates x_1 and x_2 respectively, and by Theorem III

$$P_1'P_2' = x_2 - x_1$$

furnishes the projection of the vector d on the axis OX in magnitude and sense. According to Theorem I this projection is given by $d \cos \theta$, so that withc onsiderable ease one arrives at the fundamental equations of projection:

$$\begin{aligned} x_2 - x_1 &= d \cos \theta, \\ y_2 - y_1 &= d \sin \theta. \end{aligned} \tag{1}$$

Squaring and adding we obtain the distance formula. Dividing the second of (1) by the first, we obtain the slope of the vector. If $P_3(x_3, y_3)$ divides d into segments such that

$$P_1P_3/P_1P_2 = k$$

then calling the segment P_1P_3, d_1, we may write:

$$x_3 - x_1 = d_1 \cos \theta, \qquad x_2 - x_1 = d \cos \theta.$$

Dividing, one has

$$(x_3 - x_1)/(x_2 - x_1) = k,$$

or

$$x_3 = x_1 + k(x_2 - x_1).$$

The equation of a line having a given direction and passing through a given point may be written at once from the formula furnishing the slope of the line by considering (x_2, y_2) the coördinates of any point on the line. The "normal form" for the straight line follows in a very brief way from Theorems I and II. The coördinates of any point (x, y) on the line may be interpreted as segments of a broken line, and its projection $x \cos \beta + y \sin \beta$, on p, the perpendicular from the origin, with vectorial angle β, is the projection of ρ, the join of its end points, or p itself. From this, of course, the distance from a point to a line follows. If

the equation of the line through the two points (x_2, y_2) and (x_3, y_3) be written in determinantal form, the distance h from the point (x_1, y_1) to the line is

$$h = \frac{\begin{vmatrix} x_1, y_1, 1 \\ x_2, y_2, 1 \\ x_3, y_3, 1 \end{vmatrix}}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}}.$$

Since the denominator is the base of the triangle determined by the three points, the determinantal form for the area will be evident. The transformation equations are as easily deduced by the direct application of Theorems I and II. The application of these three theorems to three dimensions is equally successful.

I do not at all mean to imply that these general proofs should be used to the exclusion of the time honored method of taking a figure in the first quadrant and deducing the result geometrically. These geometrical exercises may well be relegated to the problem lists. But the general method is the simpler, and the knowledge of the vector requisite for its use is not beyond the college freshman. For many students the vector idea can be introduced none too early. Anyone contemplating a new text on analytical geometry should certainly weigh these possibilities.

RECENT PUBLICATIONS.

REVIEWS.

Mathematical Logic.

A Survey of Symbolic Logic. By C. I. Lewis. Berkeley, University of California Press. 1918. Royal 8vo. 6 + 409 pp. Price $4.00.

Molière's M. Jourdain was very much surprised when told that he had been using prose all his life. Equally astonished are many present-day mathematicians when informed that they have been using 'logical prose'—propositional functions, the Zermelo axiom, and the like—for a correspondingly long period.

What is this logical prose of which the mathematical and the logical world at large have been, till quite recently, so blissfully ignorant? It is the principles of modern deductive logic, known also as symbolic or mathematical logic. Though Professor Lewis prefers the term *symbolic*, Russell and his school seem to have established almost irrevocably the name *mathematical* logic. And Professor Lewis's book is a survey of the history of the various stages in the discovery of the principles of deductive logic.

What are these principles? Everyone has heard of the famous 'Laws of Thought'—the Laws of Identity, Contradiction, and Excluded Middle. Assuming that these laws, considered as principles of *logic* (not of thought), are necessary, are they also sufficient? Obviously not; for the principle of the Syllogism is just as necessary to logical procedure as are these laws. Are the four principles sufficient? How shall we decide? We can study the problem empirically.

We can make a careful and detailed investigation of some of the important proofs of mathematical theorems—for instance, in function theory—and after a step-by-step analysis we may find that these proofs employ various principles in addition to the four above mentioned. Such is, for example, the principle that whenever we have three propositions p, q, and r, and we know that

(1) If p is true, then q implies r, we have a right to replace (1) by

(2) If p and q are both true, then r is true. (This principle is called by Peano the principle of Importation, since it enables us to 'import' q into the hypothesis of the main implication.) In mathematical proofs we can no more dispense with the principles of Importation than we can with the Syllogism or with the principle of Contradiction. No more can we dispense with the 'obvious' principle that whenever any two propositions p and q are both true, then propositions q and p are both true, *i.e.*, the commutativity of the logical operation called 'propositional conjunction.' And similarly with a host of other principles used implicitly by every logician and mathematician.

May not some of these principles be derivable from some of the others? Certainly. The long train of researches, culminating in the mathematical logician's 'Bible'—Whitehead and Russell's volumes of *Principia Mathematica*—has demonstrated the fact that just as the entire system of euclidean geometry can be compressed deductively into a small number of unproved geometric propositions based on a small number of undefined geometric terms, so the set of propositions that constitute the system of deductive logic can be compressed into a surprisingly small number of fundamental logical propositions based on a small number of fundamental logical terms.

But the Rome of mathematical logic was not built in a day. And the history of the long mathematical logical journey from Leibniz to Russell is a history of the triumph of the human intellect in every way as marvellous as the triumphs of the intellect in the realms of natural, as opposed to mental, experimentation.

The volume under review attempts to trace this history through its various stages. In Chapter I Professor Lewis summarizes the development of mathematical logic from Leibniz through De Morgan, Boole, Jevons, and Peirce. In Chapter II he presents the formal principles of the algebra of logic, that is, the algebra as founded by Boole and improved by Schroeder. This algebra, many of the laws of which are identical with those of number-algebras, has a 'function theory' that may well interest the student of mathematics from the standpoint of 'comparative algebra'. In Chapter III we are introduced to the well-known interpretations of this Boole-Schroeder algebra, in terms of logical classes, propositions, and relations, and also, geometrically, in terms of planar regions.[1] Ch. IV is devoted in part to an elementary and extremely successful

[1] Although Whitehead (*A Treatise on Universal Algebra*, vol. I, p. 35) characterizes this algebra as "the only known member of the non-numerical genus of Universal Algebra," mathematical readers may be interested in the following purely numerical interpretation. Let us call any positive integer, > 1, whose prime factors occur but once, a *Boolean* integer. Examples: 6, 30, 70. Choose any Boolean integer, B. Let the elements, $a, b, c, \cdots$, of this algebra be the 2^n factors of B (including 1 and B). Let $a \times b$ = H. C. F. (a, b). Then $a + b$ = L. C. M. (a, b), "0" = 1, "1" = B, "$-a$" = $B \div a$, and $a \subset b = a$ is a factor of b.'

exposition of some of the main logical doctrines of the first volume of Whitehead and Russell's *Principia Mathematica*. In Chapter V Professor Lewis outlines a system of logic differing essentially from that of Russell in the meaning assigned to the fundamentally important relation of implication. Finally, in Chapter VI the leading views on mathematical and logical *method* are compared and contrasted. In the Appendix, Professor Lewis gives a translation, from Latin, of two fragments from Leibniz on the logic of classes.

The reviewer regrets that the author has omitted all mention of the various problems of postulational technique, as well as of *Mengenlehre*—problems intimately associated with the advance of mathematical logic. But in spite of these, perhaps inevitable, omissions, Professor Lewis, in presenting for the first time the connected history of the subject, has succeeded admirably.

HENRY M. SHEFFER.

HARVARD UNIVERSITY.

College Algebra. By H. L. RIETZ and A. R. CRATHORNE. Revised Edition. New York, Henry Holt and Co., 1919. (First Edition, 1909.) 8vo. 14+268 pp. Price $1.60.

This new edition of the Rietz and Crathorne *College Algebra* differs from the first (*a*) in the addition of new exercises and the alteration of figures in old ones, (*b*) in the omission of some topics not in the main line of development, and (*c*) in a rewriting of certain sections which proved hard to teach in the original form.

The changes in the exercises are sufficient to undermine the utility of the sets of solutions which come into being on any campus when the same book has been used for several years. In some cases, in the interest of more careful grading, the order of examples has been changed and easy examples have been added. Other new examples introduce applications in fields not previously touched—in at least one case, involving 'war bread,' in a field which did not exist in 1909.

In the problems on the Theory of Equations, the authors have lost an excellent opportunity to fall in with the modern trend towards unification. Several problems (pp. 146–7) call for the depth to which a sphere of specified material will sink in water, but the necessary cubic equation is always given all 'set up.' If the student were asked to show that the depth to which a sphere of radius r and specific gravity ρ will sink is the root between 0 and $2r$ of the equation $x^3 - 3rx^2 + 4\rho r^3 = 0$, he would have to combine with the theory of equations a little solid geometry (the formula for a spherical segment), a fundamental physical definition, and a physical principle. This seems to be a striking instance of a useful bit of mathematics and physics which is not adequately treated in any course because it draws on physics and two separate mathematical subjects. Perhaps, moreover, the inclusion of this general equation would have helped the authors as well as the student, by making more likely the detection of the errors in the constant terms in Ex. 18 and 19 on page 146 in the revised edition. These errors really imply that the specific gravity of yellow pine is .9657 and of ice .693, although the first edition gives correct values, .657 and .93. It may be that this error arose from the desire not to ask (as the first edition did) for the determina-

tion of an answer to three decimal places when the given specific gravity was accurate only to two places.

Among the topics omitted is the discussion of the maximum or minimum value of the quadratic function. This omission is to be regretted if it be accepted, as the Committee on Mathematics Requirements recommends, that the choice of topics should consider primarily those who go no further in mathematics. An introduction to the mathematical discussion of maxima and minima, with some applications to problems, seems highly desirable for such students. Of course, if the book is primarily for those who will go on to the more thorough treatment possible in a calculus course, the omission is justified.

The book is improved by the omission of the detailed discussion in the first edition of the use of undetermined coefficients in expansions in series. But it would seem desirable at least to mention the fact that such coefficients are used for other purposes than partial fractions and to include a few simple examples,—such as, for instance, determining a, b, and c so that the curve $y = a + bx + cx^2$ should pass through three specified points.

Sections rewritten include the discussion of 'equivalent equations,' which in the first edition were handled too abstractly for college freshmen. The new treatment calls attention to the possibility that redundant or defective solutions may arise in the course of formal work and illustrates the point with some good simple examples. Other sections rewritten are those on mathematical induction, which have been much improved. The example worked out in the text $[1 + 3 + \cdots + (2n - 1) = n^2]$ is perhaps the simplest that can be selected in the necessary formal work, and is much more typical of the bulk of the exercises given than was the example used in the first edition. Another example is explained very carefully to show that the second part of the proof is necessary—*i.e.*, the proof that if the formula is true for $n = r$, it is true for $n = r + 1$. This example, however, is perhaps not the best possible "pathological case," as it is unlike any that the student has to prove. Perhaps a better instance would be a formula like $\Sigma n = \{n^3 + 17n - 6\}/12$, which is true for $n = 1$, 2 or 3, but not for $n = r + 1$ if true for $n = r$. Students are usually interested in the idea that in this and similar cases, untrue formulas can always be constructed which will be true for any finite number of values of n; and this fact might well be mentioned, either here or in connection with undetermined coefficients.

The treatment of probability unfortunately does not here receive the radical revision which the real importance of the subject demands. It is to be hoped that some day the course for college freshmen will include a discussion of the great extent to which probability considerations are used implicitly in ordinary thought, as well as more explicitly in statistical sciences. Perhaps the next broadening of the usual material will be a discussion of frequency curves, with the connecting argument that probability is relative frequency. The book under review, however, goes very little further than the first edition, in which probability is limited to games of chance, expectation of life, and expectation of money. The authors of this book might seem to be peculiarly the ones to make the exten-

sion proposed, on account of their statistical interests, but apparently they did not consider this revision of this book the time or place to do so.

In general the book is substantially improved and although it does not blaze out any new paths, seems well adapted to the needs of those who hold to the established system of a rather careful and extensive course in college algebra before analytic geometry and calculus are begun.

R. W. BURGESS.

BROWN UNIVERSITY,
February, 1920.

Lectures on the Theory of Plane Curves delivered to Post-Graduate Students in the University of Calcutta. By SURENDRAMOHAN GANGULI. 2 parts. Calcutta, University of Calcutta, 1919. 8vo. Part 1, 10 + 1–138b pp.; Part 2, 13 + 139–350 pp. + 17 plates.

Preface: "The present volume . . . is intended as an introductory course suitable for advanced students of geometry and assumes scarcely any further knowledge of analysis on the part of the reader than is to be found in most of the ordinary text-books on differential calculus and on analytical geometry. Throughout the whole work I have endeavored to give prominence to geometrical methods, as it appears that geometry, in judicious combination with analysis, is likely to simplify otherwise tedious and lengthy investigations. With this end in view, Professor Sylvester's Theory of residuation has been introduced at the outset and occasional application of the principle has been found very useful. I have carefully avoided complicated forms of equations but at times they have been found indispensable.

In teaching the subject constant recourse has been had to the classic treatises of Salmon and Clebsch, and the works of Basset, Scott, and others, have been frequently consulted. My obligations to these authors, which are probably much greater than I am aware of, are gratefully acknowledged. I am indebted on almost every page to the great work of Salmon on *Higher Plane Curves* and it is impossible to record in detail my obligations to this inspiring writer. The English edition of the great work of Salmon has been long out of print and it has been found necessary to publish a new text-book based on modern methods, with a view to remove the inconvenience experienced by the English-knowing students of the University."

Contents—Chapter I: Introduction, 1–10; II: Theory of plane curves, 11–32; III: Singular points on curves, 33–50; IV: Polar curves, 51–65; V: Covariant curves—the hessian, 66–81; VI: Polar reciprocal curves, 82–95; VII: Foci of curves, 96–103; VIII: The analytical triangle—asymptotes, 104–125; IX: System of curves, 126–138; X: Curves of the third order—cubic curves, 139–157; XI: Harmonic properties of cubic curves, 158–168; XII: Canonical forms, 169–203; XIII: Unicursal cubics, 204–215; XIV: Special cubics, 216–231; XV: Invariants and covariants of cubic curves, 232–243; XVI: Curves of the fourth order—quartic curves, 244–266; XVII: Trinodal quartics, 267–275; XVIII: Bicircular quartics, 276–305; XIX: Circular cubics as degenerate bicircular quartics, 306–315; XX: Special quartic curves, 316–328; Appendices, 329–345.

NOTES.

In *Historical Portraits* 1700–1850. *The Lives by C. R. L. Fletcher. The Portraits chosen by Emery Walker* (Part I, 1780–1800, Oxford, Clarendon Press, 1919) there are portraits of Newton and Halley. "Sir Isaac Newton," 29–33; opposite page 30 is a reproduction of the painting by John Vanderbank, at Trinity College, Cambridge. Edmund Halley, 33–36; opposite page 34 is a reproduction of the painting by Thomas Murray at Queen's College, Oxford.

The opening article (16 pages) of the second volume (1920) of *Norsk Matematisk Tidsskrift*, the organ of the Norwegian Mathematical Society, is a lecture on Evariste Galois delivered by the late Ludvig Sylow.

For 1920 the first A-number of *Matematisk Tidsskrift*, the organ of the Mathematical Society of Copenhagen, contains a six-page sketch by C. Juel of Hieronymus Georg Zeuthen (1839–1920) and an interesting portrait of him taken in 1880; in the first B-number is an article by T. Bonnesen on ancient and modern theories of irrationality.

Bulletin de la Société Mathématique de Gréce I, 1—Δελτίον τῆς Ἑλληνικῆς Μαθηματικῆς Ἑταιρείας, Τόμος Α΄, Τεῦχος Α΄, was published at Athens (Π. Α. Πετράκου) in May, 1919. The second number published in December completed the volume of 186 pages, rather more than a third of which is in French, the rest being in Greek. The four parts of each number are devoted to (*a*) the proceedings of the Society; (*b*) mathematical papers of a purely scientific nature; (*c*) papers in their essence philosophic, didactic, etc.; (*d*) miscellaneous mathematical news. The editors are G. Rémoundos, P. Zervos, N. Sakellarios and K. Lambiris. The scientific papers in the first number are entitled: "Formules fondamentales relatives aux courbes d'un couple de surfaces," "Les séries divergentes par le calcul des probabilités," "Sur l'équivalence des systèmes d'équations différentielles," "Sur quelques rémarques relatives aux théories de l'intégration des systèmes en involution du second ordre," "Sur les formes bilinéaires," and "Περὶ μηχανικῶν ἀναλλοιώτων."

A sumptuous volume by Mr. JAY HAMBIDGE entitled *Dynamic Symmetry, the Greek Vase*, was published last May by the Yale University Press (178 linecuts; 161 pages, small folio; price $6.00). "Dynamic symmetry deals with commensurable areas which represent the projection of solids. The symmetry of man and plant is dynamic; the symmetry of the entire fabric of classic art, including buildings, statuary and the products of all of the crafts, is dynamic. The symmetry of art since classic times is *static*.

"So revolutionary are the discoveries made by Mr. Hambidge, so tremendous will be their effect on the fundamental rules of artistic expression, that the world of art is roused to a high pitch of interest. The principles of 'dynamic symmetry',[1] have now been adopted by many craftsmen, designers and a number of important advertising illustrators." These principles are based upon golden section and the logarithmic spiral form. Professor R. C. ARCHIBALD contributes "Notes on the logarithmic spiral, golden section and the Fibonacci series," pages 146–157,—an extensive elaboration of his notes which appeared in this MONTHLY *1918*, 189–193, 232–238.

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 42, no. 1, January, 1920 [published March, 1920]: "Groups of order 2^m which contain a relatively large number of operators of order 2" by G. A. Miller, 1–10; "The Green's function for a plane contour" by H. D. Frary, 11–25; "On the solution of certain types of linear differential equations in infinitely many variables" by W. G. Simon, 27–46; "Periodic orbits on a surface of revolution," by D. Buchanan, 47–75.

[1] These principles are also developed in a monthly periodical called *The Diagonal* (Yale University Press) of which Mr. Hambidge is editor.

BROWN ALUMNI MONTHLY, Brown University, volume 20, May, 1920: "Henry Parker Manning and the development of mathematics at Brown" by R. C. Archibald, 183–185.

BULLETIN DES SCIENCES MATHÉMATIQUES, volume 44, January, 1920: Review by E. Picard of "Mémoire sur certains nombres invariants qui se présentent dans la théorie des multiplicités algébriques" by S. Lefschetz, 5–7 [From *Comptes Rendus*, volume 169, see *1920*, 143]; Review by P. Drouin of P. Boutroux's *Les Principes de l'analyse mathématique*, tome 2 (Paris, 1919), 16–20.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 26, no. 6, March, 1920: "The twenty-sixth annual meeting of the American Mathematical Society" by F. N. Cole, 241–259; "The St. Louis meeting of the American Mathematical Society" by O. D. Kellogg and A. Dresden, 260–273; "Poncelet polygons in higher space" by A. A. Bennett, 274–275; "On the rectifiability of a twisted cubic" by Mary F. Curtis, 275–277; "Note on linear differential equations of the fourth order whose solutions satisfy a homogeneous quadratic identity" by C. N. Reynolds, Jr., 277–280; "An acknowledgment of priority" by A. A. Bennett, 280–281; "Dickson's *History of the Theory of Numbers*" by D. N. Lehmer, 281; "Notes," 281–285; "New publications," 285–288.

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY, volume 10, no. 3, December, 1919: "On a special square matrix of order six" by C. E. Cullis, 127–140; "On the formation of optical images by a diffracting boundary" by B. C. Das, 141–150; "On Joachimsthal's attraction problem" by S. C. Dhar, 151–156; "On the potentials of heterogeneous incomplete ellipsoids and elliptic discs" by N. Sen, 157–178; "On the wave-equation in ellipsoidal coördinates" by S. Banerji, 179–186; "On the numerical calculation of the roots of the equations $P_n^m(\mu) = 0$ and $\frac{dP_n^m(\mu)}{d\mu} = 0$ regarded as equations in n," Part 2, by B. Pal, 187–194.

HIBBERT JOURNAL, London, volume 18, no. 3. April, 1920: "Euclid, Newton, and Einstein" by C. D. Broad, 425–458 [Last paragraph: "I have now fulfilled my promise to the best of my ability. We have seen what exactly Einstein's theory is and how it is related to Euclidean geometry and to Newtonian mechanics. The connection with the former is not really very intimate, and Einstein himself makes very little play with it. The connection with the latter is all-important. Einstein's discovery synthesizes Newton's two great principles—the laws of motion and the law of gravitation. It removes the obscurity that has always hung over the former, by working out the relativity of motion to the bitter end, whilst it generalizes and slightly corrects the latter and accounts for its peculiar position among all the other laws of nature. Such work can only be done by a man of the highest scientific genius, and we have no right and no need to enhance his greatness by decrying the immortal achievements of his predecessors. It is enough that we can, without the slightest flattery or hyperbole, class Einstein with Newton, and say of the former what is written on the tomb of the latter:—'Sibi gratulentur homines tale tantumque exstitisse humani generis decus.'"]

JOURNAL OF EDUCATION, Boston, volume 90, December 25, 1919: "Concrete geometry for seventh grade" by W. H. Fletcher, 654–657.

MATHEMATICAL GAZETTE, volume 10, March, 1920: The annual meeting of the Mathematical Association, 17–19; "Gleanings far and near," 19, 29, 34; "Geometry teaching: the next step" by C. Godfrey, 20–24; "Convention and duplexity in elementary mathematics" by E. H. Neville, 25–26; "The position of common logarithms in mathematical training," by H. M. Cook, 27–28 [followed by discussion, 28–29]; "The teaching of mechanics to beginners" by R. C. Fawdry, 30–34; "The graphical treatment of differential equations" (continued) by S. Brodetsky, 35–38; "Coördinate geometry in schools" by W. J. Dobbs, 39–41; "Obituary, G. W. Palmer," 42 [Mr. Palmer's death was recorded in the MONTHLY, *1920*, 43. Quotations: "We have heard it said that he well deserved to be called the 'Father of Arithmetic' in English education. . . . By the stress we have laid on his contributions to the literature of one branch of elementary mathematics, we do not wish to imply that his work was thereby limited. It may be said, indeed, that he inaugurated a new era in the teaching of mathematics at Christ's Hospital]; "Mathematical Notes" by G. H. Bryan ('A formal geometrical construction for the solution of the sound ranging problem'), A. O. P. ('A curiosity'), T. Carleman (Chances in winning a game at lawn tennis), and W. E. H. Berwick ('The four fours'), 43 [The curiosity is: $\frac{1}{3}\frac{8}{6}\frac{5}{2}\frac{3}{7}\frac{4}{8} \times \frac{1}{5}\frac{7}{9}\frac{4}{8}\frac{6}{2}\frac{9}{3} = \frac{3}{5}\frac{4}{0}\frac{1}{9}\frac{8}{7}\frac{2}{6}$. Here the set of digits occurs in each fraction, each digit once and only once]; Reviews and notices, books received, contents of journals, etc., index to volume 9, 44–48 + 14 pp.

MESSENGER OF MATHEMATICS, volume 49, no. 3, July, 1919: "Factorisation of N & $N' = (x^n \mp y^n) \div (x \mp y)$, &c. [when $x - y = 1$]" (continued) by A. Cunningham, 33–36; "The eliminant of two binary quantics with determinantal coefficients" by T. Muir, 37–41; "On certain plane configurations of points and lines" by W. Burnside, 41–43; "A property of groups of even order" by W. Burnside, 43; "On the solution of a cubic equation" by A. Lodge, 44–48. No. 4, August: "On the solution of a cubic equation" (continued) by A. Lodge, 48–51; "On uniform Diophantine approximation" by H. T. J. Norton, 51–57; "Standard relation of Legendre's functions" by R. Hargreaves, 58–62; "Note on the mth compound of a determinant of the $(2m)$th order" by T. Muir, 62–64.

NATURE, volume 105, March 18, 1920: "Mathematics: pure and applied" by S. Brodetsky, 64–67 [reviews of F. Slate's *The Fundamental Equations of Dynamics and its Main Co-ordinate Systems Vectorially Treated and Illustrated from Rigid Dynamics* (Berkeley, 1918), of L. Silberstein's *Projective Vector Algebra: An Algebra of Vectors Independent of the Axioms of Congruence and of Parallels* (London, 1919), of E. S. Andrews's *Elements of Graphic Dynamics* (London, 1919), of C. Davison's *Differential Calculus for Colleges and Secondary Schools* (London, 1919) and of J. Milne's *The Analytical Geometry of the Straight Line and the Circle* (London, 1919)]; "Some methods of approximate integration and of computing areas" by A. C. Percival, 70–71; "Time-reckoning of the North American Indians," 75; "The gyrostatic compass" by S. G. Brown, 77–80.—March 25: "Aeronautical research" 95–97 [Review of L. Bairstow's *Applied Aerodynamics* (London, 1920).]—April 1; "Some methods of approximate integration and of computing areas" by R. A. P. Rogers, 138 ["The formulæ which Mr. Percival gives in *Nature* for March 18 for approximate integration are well known, but there are one or two points in connection with them which are frequently overlooked, especially by writers of books on mathematics for engineers . . ."]; "Gravitational deflection of high-speed particles" by H. G. Forder, 138 ["The result mentioned by Mr. Leigh Page and verified by Prof. Eddington (*Nature*, March 11, p. 37), that the gravitational effect on a particle travelling radially is a repulsion if the speed exceeds $11\sqrt{3}$ times the light-velocity, is given by Hilbert in the *Göttinger Nachrichten* for 1917. The same paper contains interesting remarks on the path of a particle or light-pulse moving spirally round the gravitation centre."]—April 8: "Recent mathematical books" by J. M., 162–163 [review of Karpinski, Benedict, and Calhoun's *Unified Mathematics* (Boston, 1918), C. H. P. Mayo's *Elementary Calculus* (London, 1919), J. W. Angles's *Mensuration for marine and mechanical engineers* (London, 1919), W. G. Borchardt's *School Mechanics*, Part 1. *School Statics* (London, 1919).]—April 15: "Matrices" by G. B. M., 191–192 [review of C. E. Cullis's *Matrices and Determinoids* (Cambridge, 1918)]; "A dynamical specification of the motion of Mercury" by G. W. Walker, 198–199.—April 22: "Gravitational deflection of high-speed particles" by L. Page, 233.—April 29: "Critical mathematics" by G. B. M., 256–267 [review of P. Boutroux's *Les principes de l'analyse mathématique; exposé historique et critique* (Paris, 1919)]; "Artillery science" by G. Greenhill, 268–270; "Courses on the history of science," 279.—May 6: "Euclid's Elements" by G. B. M., 288–289 [review of T. L. Heath's *Euclid in Greek, Book I* (Cambridge, 1920)]; "Leonardo de Vinci" by E. McCurdy, 307–309.—June 17: Review of W. W. Smith's *A Theory of the Mechanism of Survival: The Fourth Dimension and its Applications* (London, Kegan, Paul, Trench, 1920), 484; "A new method for approximate evaluation of definite integrals between finite limits" by T. Y. Baker, 486; "S. Ramanujan, F.R.S." by G. H. Hardy, 494–495 [Quotations: "I first heard of Ramanujan in 1913. The first letter which he sent me was certainly the most remarkable that I have ever received. There was a short personal introduction written, as he told me later, by a friend. The body of the letter consisted of the enunciations of a hundred or more mathematical theorems. Some of the formulæ were familiar, and others seemed scarcely possible to believe. A few (concerning the distribution of primes) could be said to be definitely false. There were no proofs, and the explanations were often inadequate. . . . Whatever reservations had to be made, one thing was obvious, that the writer was a mathematician of the highest quality, a man of altogether exceptional originality and power. . . . Ramanujan's activities lay primarily in fields known only to a small minority even among pure mathematicians—the applications of elliptic functions to the theory of numbers, the theory of continued fractions, and perhaps above all the theory of partitions. His insight into formulæ was quite amazing, and altogether beyond anything I have met with in any European mathematician. It is perhaps useless to speculate as to his history had he been introduced to modern ideas and methods at sixteen instead of at twenty-six, It is not extravagant to suppose that he might have become the greatest mathematician of his time. What he did actually is wonderful enough. Twenty years hence, when the researches which his work has suggested have been completed, it will probably seem a good deal more wonderful than it does to-day."]

NOUVELLES ANNALES DE MATHÉMATIQUES, volume 78, December, 1919: "Avis" by les rédacteurs, 441–443 [Quotations: ". . . Nous avons ainsi été conduits à concevoir une véritable réorganisation de ce journal, dans le sens que nous allons indiquer. Chaque numéro des *Nouvelles Annales* contiendra, en principe, des *Mémoires originaux*, des *Exercices de Licence et d'Agrégation*, une *Chronique du mouvement mathématique*, des *Enonces et des solutions de questions* . . . Dans la *chronique* . . . nous publierons des nouvelles intéressant le monde des mathématiciens, telles que nominations, distinctions, ouvertures de cours importants, des résumés plus ou moins détaillés de découvertes récentes (sans qu'il s'agisse d'un dépouillement systématique des périodiques, travail pour lequel il exist des publications auquelles les *Nouvelles Annales* ne prétendent pas se substituer); des analyses bibliographiques, etc."]; "Sur les équations de Didon" by P. Humbert, 443–451; "Sur le cercle de Miquel" by F. Girault, 452–456; "Sur les surfaces tétraédrales symétriques" by C. Servais, 456–468; Questions and solutions, 468–472; Index, 473–480.—Volume 79, January, 1920: "Exposé élémentaire d'une théorie rigoureuse des liaisons finies unilatérales" by E. Delassus, 1–12; "Simple remarque sur la cyclide de Dupin" by M. d'Ocagne, 13–14; "Lieux des foyers ordinaires des courbes algébriques d'un faisceau tangentiel ou ponctuel" by T. Lemoyne, 14–17; Licence questions 17–30; Chronique, 31–35; Questions and solutions, 35–40.

PEDAGOGICAL SEMINARY, volume 26, no. 4, December, 1919: "Relation of initial ability to the extent of improvement in certain mathematical traits" by F. M. Phillips, 330–355.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY OF LONDON, series A, vol. 220, no. A579, April 27, 1920: "A determination of the deflection of light by the sun's gravitational field, from observations made at the total eclipse of May 29, 1919" by F. W. Dyson, A. S. Eddington, and C. Davidson, 291–333 + plate 1.

PROCEEDINGS OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES, volume 55, no. 3, March, 1920: "Contribution to the general kinetics of material transformations" by A. J. Lotka, 135–154.—No. 4, March: Rotations in space of even dimensions" by H. B. Phillips and C. L. E. Moore, 155–188.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 6, no. 2, February, 1920: "Groups generated by two operators, S_1, S_2, which satisfy the conditions $S_1^m = S_2^n$, $(S_1S_2) = 1$, $S_1S_2 = S_2S_1$" by G. A. Miller, 70–73; "The larger opportunities for research on the relations of solar and terrestrial radiation" by C. G. Abbott.—March: "Note on geometrical products" by C. L. E. Moore and H. B. Phillips, 155–158.

PROCEEDINGS OF THE ROYAL SOCIETY, volume 97, no. A683, April 15, 1920: "A new apparatus for drawing conic curves" by A. F. Dufton, 199–201 [Quotations: "1. The attention of mathematicians has been attracted to the mechanical description of conic sections since the discovery of the curves by Menaechmus, but in the numerous mechanisms which have been invented only partial success has been attained.

"In an early conograph, the invention of which is ascribed to the Arabs, the curve is the actual intersection of the surface upon which it is drawn with a straight line generating a cone. Instruments of this kind were designed at the end of the sixteenth century for use in the construction of sun dials.

"Newton in his 'Principia' (Lib. I, Prob. XIV) discusses the drawing of conic sections and describes a mechanical method of plotting them. Two angles are rotated about their vertices and the intersection of one pair of arms is kept upon a fixed straight line. The locus of intersection of the other pair is a conic section.

"Sylvester (*Proc. Roy. Inst.*, vol. 7, p. 179 (1873–75)) showed that a conic can be drawn by means of an apparatus of thirteen links. His method fails to draw the curve at the vertex but is simpler than that of Peaucellier (ibid.), which involves the use of fifteen links besides a cross-piece rigidly attached to one of them. Both these methods depend upon the principle of inversion.

"A conograph based upon the constancy of the anharmonic ratio subtended at the tracing point by four fixed points on the curve was invented by Willy Jürges (*Zeitschrift für Math. und Physik*, vol. 38, p. 350 (1893)). In this instrument the use of eight sliding constraints makes smooth work difficult.

"2. With the apparatus described in this paper, the conic is drawn as the polar reciprocal of a circle. . . . In the remarkable precision of even a roughly made instrument, in the tracing of the curve at one sweep and in the application to all conics from circle to straight line, the apparatus offer a satisfactory solution to a very ancient problem".

QUEEN'S QUARTERLY, Queen's University, volume 27, no. 3, January, 1920: "Relativity and gravitation," part 1, by E. Flammer, 286–308.—April: "Relativity and gravitation," part 2, by E. Flammer, 424–442.

SCHOOL SCIENCE AND MATHEMATICS, volume 20, no. 3, March, 1920: "Some problems for the class room from the orientation work of the A. E. F." by C. A. Epperson, 210–213; "Problem department," 270–272.—No. 4, April, 1920: "Some factors affecting the selection of the high-school course of study and methods of teaching mathematics" by H. R. Douglas, 287–299; "Second list of marginal notes on Cajori's history of mathematics" by G. A. Miller, 300–304; "Up-to-date problems in junior high school mathematics" by T. Lindquist, 305–311; Problems and solutions, 365–368, 374; "Meeting of Association of Teachers of Secondary Mathematics in North Carolina," 378; "The National Council of Mathematics Teachers," 380–382.

SCIENCE, n.s., volume 51, April 2, 1920: "Notice of a recent contribution to statistical methods" by G. F. McEwen and E. L. Michael, 349–350 [Compare this MONTHLY, *1920*, 133].—April 9: "Concerning ballistics" by A. G. Webster, 368–369.—April 30: "Unification of symbols and diagrams" by W. P. White, 436–437 [First paragraph: "The recent attempts to unify the mathematical symbols used in physics and chemistry are probably approved, in principle, by practically everyone. They have stimulated and guided a large amount of voluntary effort and coöperation. Their complete recognition and adoption has been hindered by the difficulty of getting any one system to satisfy the varied requirements and personal preferences involved."] —May 7: "Modern interpretations of differentials" by A. S. Hathaway, 464–465 ["To the Editor of *Science:* Professor E. V. Huntington, in an article entitled 'Modern interpretation of differentials' (*Science*, March 26), states with reference to the $\lim \Delta y = 0$, $\lim N\Delta y = dy$, that, 'The inevitable consequence of such a definition is that $dy = 0$, which is futile.' Every school boy in the theory of limits knows that this is not true when N varies . . ."]; Review by R. von Huhn of A. C. Haskell's *How to make and use graphic charts* (New York, 1919), 466–467 [Cf. *1920*, 269–270].—May 21: "Formulæ giving the day of the week of any date" by W. J. Spillman, 513–514; "A new statistical journal" by R. Pearl, 515–517 [*Metron*, Tipografia Industrie grafiche Italiane, via Viscovado, Padova, Italy; quarterly, 40 lire a year]; "The American Mathematical Society", by F. N. Cole, 523–524.—May 28: "The Mathematical Institute at the University of Strasbourg," 534.—June 11: "Modern interpretation of differentials again" by E. V. Huntington, 593.—June 18: "Aristotle and Galileo on falling bodies" by F. Cajori, 615–618.—Volume 52, July 9: "Modern interpretation of differentials" by A. S. Hathaway, 35.

TEXAS MATHEMATICS TEACHERS' BULLETIN, volume 5, no. 1, November, 1919: "Freshmen mathematics prizes" by H. J. Ettlinger, 5–6; "Mathematics Section, State Teachers Association" by Goldie P. Horton, 7–8; "The Pythagorean Theorem" by P. M. Batchelder, 9–13; "The quadratic equation and its solution" by J. W. Calhoun, 14–18; "On the disciplinary and applied values of mathematical study" by C. N. Moore, 19–26 [Reprinted from *Education*, December, 1918; cf. *1919*, 162]; "Practical figures in solid geometry" by Elizabeth Dice, 27–29; "Some problems in applied plane geometry" by H. J. Ettlinger, 30–32; "Topics and references for the high school mathematics club" by Goldie P. Horton, 33–34; "The straight edge" by A. N. Onymous, 35—No. 2, February, 1920: "Mathematics in the summer school" by H. J. Ettlinger, 5; "A report on the teaching of high school mathematics" by H. J. Ettlinger, 6–7; "Asking questions" by T. McN. Simpson, 8–11; "Limit proofs in geometry" by A. A. Bennett, 12–21; "What is mathematics?" by P. M. Batchelder, 22–24; "Sundials" by Mary E. Decherd, 25–30; "Mathematics at the Houston meeting of the State Teachers Association" by Goldie P. Horton, 31–32; "The straight edge."

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 21, no. 2, April 1920: "Differential equations containing arbitrary functions" by G. A. Bliss, 79–92; "Functions of lines in ballistics" by G. A. Bliss, 93–106; "On the summability of the developments in Bessel's functions" by C. N. Moore, 107–156; "One parameter families and nets of ruled surfaces and a new theory of congruences" by E. J. Wilczynski, 167–206; "Note on space curves" by G. M. Green, 207–236 ["The manuscript of this paper was found among the papers in the late Dr. Green's handwriting which were turned over to me for investigation. It was complete except for a few references, which I have supplied; the only changes which I have permitted myself to make are concerned with two or three places where the language seemed to be ambiguous. This paper constitutes a notable addition to Green's writings, and shows once more the fertility of his imagination and his unfailing ability to see something new even in the most familiar fields.—E. J. Wilczynski"]; "A set of postulates for fields" by N. Wiener, 237–246; "A theorem on modular covariants" by Olive C. Hazlett, 247–254.

UNIVERSITY OF CALIFORNIA PUBLICATIONS IN MATHEMATICS, volume 1, no. 12, April 12, 1920: "A set of five postulates for Boolean algebras in terms of the operation 'exception'" by J. S. Taylor, 241–248.

ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT, volume 50, nos. 11–12, published January 10, 1920: "Zur elementaren Behandlung von Exponentialfunktion und Logarithmus" by A. Loewy, 330–339; "Bücherbesprechungen," 340–347—Volume 51, no. 1, published February 5: "Die Simsonsche Gerade" by R. Henke, 1–12; Lektorate für Mathematik, ein Vorschlag zur Erweiterung des mathematischen Hochschulunterrichts" by A. Rohrberg, 13–17; "Kleine Mitteilungen," 18–24.

AMERICAN DOCTORAL DISSERTATIONS.

S. P. SHUGERT, The resolvents of König and other types of symmetric functions. Lancaster, Pa., 1919. 19 pp. (University of Pennsylvania, 1914.)

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

NOTES AND SUGGESTIONS.

If our readers know of the formation or existence of undergraduate mathematics clubs which have not been mentioned in this department we would be glad to have them send us full information concerning their organization and activities. We hope, too, that clubs which plan their programs for the year soon after the opening of college will send us copies without waiting for a request from the editor. The issuing of a printed folder in October or November giving information about the club as well as times, places and programs for meetings during the year seems to contribute decidedly to the success of the work and the number of clubs which are adopting the plan is increasing. A sample of such a folder will be sent by the editor of the department to any one requesting it.

Appendix A, pages 227–230 of Marie Gugle's *Modern Junior Mathematics*, Book 2 (New York, The Gregg Publishing Co., 1920), is devoted to "Mathematics Clubs." Sample programs are given to show how the same topic may be used in various grades by making the treatment different in each case. A list of thirty-four topics and twelve titles of books and magazines are given. While these programs, topics and references are intended for high schools they may prove suggestive to club program makers in colleges.

In response to our request published sometime ago for suggestions which might be used in social meetings, Mr. John W. Arnold, Treasurer of the Undergraduate Mathematics Club at the University of Illinois, sends the following account of the annual social meeting of their club held Friday evening, March 12, 1920.

The company of about eighty persons was numbered and tagged upon arrival and later divided by means of the last digit of the number into five groups—followers of Newton, Descartes, Euclid, Leibniz, and Pythagoras—who arranged

themselves around separate tables and vied with each other for the honors of the evening. As a beginning, the several parties, by turns, presented charades representing mathematical terms which were guessed with more or less difficulty by the other groups. Music followed and then some time was spent at work on a motley assortment of odd, unusual, and unique problems of a generally baffling nature. All the old and a number of new tricks, puzzles and paradoxes were brought out and debated within the groups, solutions being presented to "headquarters" for verification. For the remainder of the evening the gathering puzzled over mathematical anagrams. "Geometrical refreshments" were served in the form of gelid spheres inscribed in right circular cones in quantities approaching ∞.

CLUB ACTIVITIES.

MATHEMATICS CLUB OF THE UNIVERSITY OF NEBRASKA, Lincoln. [*1918*, 313, 459; *1919*, 263–264.]

The officers of the club for the year 1919–20 were: President, Josiah A. Brooks '20; vice-president, Stella Abraham '21; secretary, Benjamin F. Margolin '21.

October 27, 1919: "The classification of real numbers" by Professor William C. Brenke.

November 13: "Curious configurations in geometry" by Professor Tracy A. Pierce; "Binomial coefficients" by Randolph T. Major '22.

January 16, 1920: "The Rhind mathematical papyrus" by Professor Albert L. Candy; "Formulæ used in artillery" by George S. Madsen '22; "How long a strip of paper would be required to write out the value of $2^{2^{36}}$, counting 10 digits per inch?", solution by Thomas G. Bowman '22.

February 12: "Horner's method of approximating roots" by Thomas G. Bowman '22; "The quadratrix" by Stella Abraham '21.

March 11: Social meeting. Professor Mayer G. Gaba gave a humorous reading "The new education in mathematics" and Miss Esther Daily furnished music. Mathematical games were played and refreshments served.

PI MU EPSILON, MATHEMATICAL FRATERNITY, OHIO STATE CHAPTER, Columbus, O.

This chapter formed a tentative organization in June and received its charter in October of 1919.

Readers of the MONTHLY will recall the account already given (*1918*, 271) of the Pi Mu Epsilon mathematical fraternity organized (1913) and incorporated (1914) at Syracuse, N. Y. The Ohio State chapter holds its charter from Syracuse and, like the parent chapter, it "aims to promote mathematics and scholarship." The high standards for eligibility to membership inaugurated at Syracuse have been maintained at the Ohio State University.

The active membership now numbers 39, of whom 18 are faculty members and graduate students and 21 are senior undergraduates. The list of charter members included every member of the mathematics department faculty.

The founding of the honorary mathematical fraternity was welcomed with much enthusiasm by the advanced students of mathematics. Departmental fraternities, both active and purely honorary, are numerous at Ohio State and they appear to meet a real need. Any college favored with an active chapter of Sigma Xi will readily understand the advantages of coöperative effort. Pi Mu Epsilon hopes to accomplish for the more limited field of mathematics what Sigma Xi is doing in the larger field of science. Ohio State has every intention of being an active chapter.

The officers for the year 1919–20 were: Director, Professor C. L. Arnold; vice-director, Professor Grace M. Bareis; secretary, Clarice S. Hobensack Gr.; treasurer, J. R. Anderson '20.

The following papers have been presented: "Differential symbolism in connection with series" by Professor R. D. Bohannan; "Three remarkable fractions appearing in a parameter study of certain curves of order ∞^3" by Professor C. L. Arnold; "A graphical presentation of the locus of the perihelion and aphelion points of certain satellites" by Harry F. Kohl Gr.; "Functions having preassigned zeros and infinities" by Mabel M. Madden '20; "A generalization of the law of signs" by Clarice S. Hobensack Gr.

Ohio State has, also, a strong undergraduate mathematics club which meets twice a month with an average attendance of over fifty. The officers of this club are all undergraduates and its papers and discussions lie in the field of interest of its younger members. Speaking of the two organizations our correspondent, Professor C. L. Arnold, says:

"There is a need for both the club and the fraternity. They coöperate rather than conflict. Mathematics is an elective in all the general courses. The rôle of the club is to foster interest early in the student's elective course. Pi Mu Epsilon hopes to recognize and honor exceptional work at the end of the course. Other departments do this. Surely the mathematician deserves and values an equal recognition."

THE PENTAGRAM, University of Texas, Austin. [*1918*, 273–274; *1919*, 364.]

The following programs have been given during 1919–20.

October 22, 1919: "Bees" by Professor M. B. Porter.

November 5: "Some ruler constructions" by Edison H. Thomas '20.

November 25: Social meeting at the home of Professor J. W. Calhoun.

December 3: "Mathematics and mental development" by Mary F. Decherd, Instructor in mathematics.

January 14, 1920: "Bonds" by Henry H. Hammer Gr.

January 28: "Some definite integrals by actual summation" by Joseph E. Burman Gr.

February 11: "Line coördinates" by Martha Randall '20.

March 3: "Codes and ciphers" by Renke G. Lubben '21; "Egyptian hieroglyphics" by Professor M. B. Porter.

March 10: Party at the home of Miss Goldie P. Horton, Instructor in pure mathematics, Misses Horton and Decherd hostesses.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

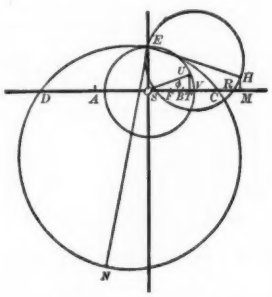
Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

A CONSTRUCTION OF THE REGULAR POLYGON OF SEVENTEEN SIDES.[1]

BY L. LINN SMITH, Student at Grinnell College.

Many methods have been devised for constructing the regular 17-gon with ruler and compasses. These are based on the fact, as demonstrated by Gauss, that the solution of the equation, $x^{17} - 1 = 0$, can be reduced to the solution of quadratics. This reduction is clearly discussed in the little book, Klein's *Famous Problems in Elementary Geometry*.[2] After removing the factor $(x - 1)$ the equation is reduced to the four following quadratics: $z_1 = 2\cos 2\pi/17$ is a root of

(1) $\quad z^2 - y_1 z + y_4 = 0 \qquad$ where $z_1 > z_2$.



Here y_1 is the larger root of the equation

(2) $\quad y^2 - x_1 y - 1 = 0$

and y_4 is the larger root of the equation

(3) $\quad y^2 - x_2 y - 1 = 0;$

while x_1 and x_2 are the roots of the equation

(4) $\quad x^2 + x - 4 = 0 \qquad$ where $x_1 > x_2$.

We now obtain the required root z_1 by constructing the roots of these various equations. As a basis we use the method of constructing a quadratic's roots where, considering the equation of the form

$$x^2 - ax + b = 0,$$

a circle is erected on the line joining the points (0, 1) and (a, b) as a diameter and the roots are the abscissas of the two intersections of the circle and the X axis.

In the figure, E is the point (0, 1) and N is the point $(-1, -4)$ and thus the circle on the line EN as a diameter solves the equation (4) and gives us

$$OC = x_1 \quad \text{and} \quad OD = x_2.$$

A and B are the mid-points of OD and OC respectively. F is the intersection of the circle, whose center is A and radius AE, with the X axis, and thus gives us the larger root of equation (3)

$$OF = y_4.$$

[1] This article and the next one should be considered with problem 2745 below.

[2] Translated into English by Beman and Smith, Boston, 1897, pp. 24–32.

The reason for this is that the circle ordinarily used to construct these roots is symmetrical with the X axis and thus we find the center by the simple bisection and do not have to draw this circle to find the one root that we need from this equation. In exactly the same manner M is the intersection of the X-axis and the circle with center B and radius BE. Hence we have the larger root of the equation (2),

$$OM = y_1.$$

Now erect a perpendicular to the X axis at M and lay off

$$MH = OF.$$

Thus the coördinates of the point H are (y_1, y_4). It is evident that the circle on EH as a diameter intersects the X axis so that OS and OR are the roots of equation (1) and since OR is the larger,

$$OR = 2\cos\frac{2\pi}{17}.$$

Having twice the cosine of the angle it is an easy matter to find the required angle. Bisect OR in T and draw the unit circle VUE. Construct the angle, $\varphi = 2\pi/17$ from the known cosine. Of course UV subtending the angle φ is the required side of the regular 17-gon.

Gauss and the Regular Polygon of Seventeen Sides.

By R. C. Archibald, Brown University.

Recent publications call attention to new material[1] in connection with the history of the construction of the regular polygon of seventeen sides. The discovery that this construction could be effected with ruler and compasses only, was one of which Gauss was vastly proud[2] throughout his life and also, according to Sartorious von Waltershausen,[3] the one which decided him to dedicate his life to the study of mathematics. As it is recorded of Archimedes that he desired a sphere inscribed in a cylinder to be engraved on his tombstone, and similarly with Ludolph van Ceulen as to the value of π to 35 places of decimals, and with

[1] *C. F. Gauss als Geometer* von P. Stäckel (*Materialien für eine wissenschaftliche Biographie von Gauss*, Heft V), Leipzig, 1918 (see pp. 78, 96); *Carl Friedrich Gauss Werke*, Band 10, Abteilung 1, Leipzig, 1917 (see pp. 3–4, 120–126, 487 and the facsimile of Gauss's notebook 1796–1814).

[2] The very first entry in his notebook "1796 Mart. 30–1814 Jul. 9" is: "1796. Principia quibus innitur sectio circuli ac divisibilitas eiusdem geometrica in septemdecim partes etc. Mart. 30. Brunsv[igae]." Again, in his own copy of his *Disquisitiones Arithmeticae* he wrote the following note in the margin beside article 365: "Circulum in 17 partes divisibilem esse geometrice, detexi-mus 1796 Mart. 30." And finally, on page 77 of "Scheda Af," begun in 1801, Gauss brought together a number of dates which were of importance to him. The first four of these were: 1. Jan., 1801, Ceres discovered; 28. March, 1802 Pallas discovered; 19. Feb., 1803 Pallas rediscovered; 30. March, 1796, "Construction des 17 Ecks."

[3] *Gauss zum Gedächtnis*, Leipzig, 1856, p. 16: "Diese Entdeckung, welche Gauss bis zum Ende seines Lebens sehr hoch schätzte, ist es vornehmlich gewesen, welche seinem Leben eine bestimmte Richtung gegeben hat, denn von jenem Tage an war er fest entschlossen, nur der Mathematik sein Leben zu widmen."

Jacques Bernoulli as to a logarithmic spiral, so also, according to Weber,[1] Gauss requested that the regular polygon of seventeen sides should be engraved on his tombstone.[2]

In a letter to Gerling dated January 6, 1819, Gauss elaborates the thought underlying the theory of polygons in his *Disquisitiones Arithmeticæ* (*D.A.*), Lipsiæ, 1801, through special reference to the polygon of seventeen sides; then after a paragraph generalizing the result to suitable prime numbers $p = 3, 5, 17, 257, 65537, \ldots$ he continues:

"The history of this discovery has up to the present nowhere been publicly alluded to by me; I can give it very exactly, however. The day was the March 29, 1796,[3] and chance had absolutely nothing to do with it. Before this, indeed during the winter of 1796 (my first semester in Göttingen), I had already discovered everything related to the separation of the roots of the equation

$$\frac{x^p - 1}{x - 1} = 0$$

into *two* groups, on which the beautiful theorem on the lower part of page 637[4] depends, without making note of the day. After intensive consideration of the relation of all the roots to one another on arithmetical grounds, I succeeded during a holiday in Braunschweig, on the morning of the day alluded to (before I had got out of bed), in viewing this relation in the clearest way, so that I could immediately make special application to the 17-side and to the numerical verification. Of course still other investigations of the seventh section of the *D.A.* were added later. I announced this discovery in the *Literaturzeitung* of Jena where my advertisement was published in May or June 1796.[5] The printing of my *Disq. Arith.* began in April 1798, continued slowly and was several times entirely stopped (because the printer moved away from Braunschweig, whence from sheet *R* on another type is used) and was completed in the summer of 1801.

"In the year 1798 or 1799 v. Zimmerman told something of my research to a certain Hugene or Huguenot (a Prussian officer) passing through Braunschweig. On his request I gave him a little paper, which he kept, containing the complete theory of the 17-side (about as above, but much more in detail). This person had the impudence afterwards to publish a work,[6] which I have not myself seen,

[1] *Encyclopädie der elementaren Algebra und Analysis* bearbeitet von H. Weber. Zweite Auflage, Leipzig, 1906, p. 362.

[2] This request was not granted, as it was in each of the other cases mentioned.

[3] At this time Gauss was eighteen years of age, but was nineteen on the following April 30.

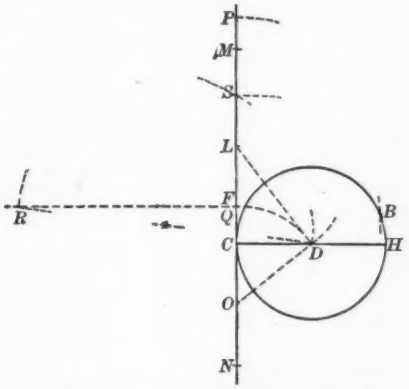
[4] Gauss, *Werke*, Band I, p. 443.

[5] *Intelligenzblatt* of the *Allgemeine Literatur-Zeitung*, Nr. 66, 1 Junius, 1796, col. 554. Two mistakes in connection with this reference are made by Klein, "Gauss' Wissenschaftliches Tagebuch 1796–1814 mit Anmerkungen herausgegehen" (*Mathematische Annalen*, Band 57, 1903); his reproduction of Gauss's advertisement is also inexact. Gauss's "Tagebuch" is reprinted in facsimile in *Gauss Werke*, Band 10. The last sentence of Gauss's advertisement is as follows: "Diese Entdeckung ist eigentlich nur ein Corollarium einer noch nicht ganz vollendeten Theorie von grösserem Umfange, und sie soll, sobald diese ihre Vollendung erhalten hat, dem Publicum vorgelegt werden."

[6] *Mathematische Beyträge Zur Weiteren Ausbildung angehender Geometer*, von dem K. Preuss. Hauptmann im Feld-Artillerie Corps v. Huguenin, Königsberg, bey Goebbels & Unger, 1803.

where he sets forth this matter in such a way that the reader might well believe that it was Huguenot's own work! Unluckily however this book was first printed several years after the appearance of my *D.A.*, so that his shamelessness is all the more ridiculous."[1]

In a letter to Gauss dated Helmstedt, March 22, 1802, and first published in 1917 (l.c.), Professor J. F. Pfaff quoted the following construction for the 17-side from a letter he had received from Professor Pfleiderer: "Diametro CH circuli dati ad extremum eius C normalis ducatur PN recta: a qua abscindantur primum $CF = \frac{1}{4}CH$; tum FM, FN utraque $= FH$. Bifariam in punctis L, O secentur rectae CM, CN: et ad centrum D circuli propositi ductis LD, OD rectis, ab perpendiculo PN abscindantur $LP = LD$, $OQ = OD$. Ad rectam CQ applicetur rectangulum (CSQ) aequale rectangulo sub DC et SP, et excedens quadrato (quod fit, si recta PN ad Q punctum normalis ducitur $QR = CP$, et super diametro DR circulus describitur secans rectum CP in S). Denique centro C, intervallo CS, describatur circulus, secans propositum circulum in B. Erit HB latus polygoni regularis XVII laterum, circulo dato inscribendi."



Gauss refers further to the regular polygon of seventeen sides in a report of a paper delivered by Erchinger in 1825 before the Royal Society of Sciences at Göttingen.[2] Gauss gives Erchinger's geometrical construction[3] of the 17-side and remarks that it flows naturally from equations which he had given in the *Disquisitiones*. He then points out that the merit of Erchinger's paper was not so much in this construction as in the synthetic proof[4] of its correctness since "this is carried through with such admirable painstaking care to avoid anything not elementary, that it reflects honor on the author and inspires the hope, that his truly uncommon mathematical talent may find every encouragement."

[1] Gauss was inaccurately informed as to the book of v. Hugenius. On page 283 of this book is the following footnote: "That the regular seventeen-sided polygon inscribed in a circle could be constructed by quadratic equations, was first discovered by a young geometer of Braunschweig whose name as I remember it is Gauss: he made the results known to several scholars and since this aroused wonderment and perhaps some disbelief, he gave also something of his method. At this time (in the year 1796) I happened to be in Braunschweig where apart from all that one knew of this unexpected solution, I had the pleasure in a short time, on following this trail, of eliciting *his own solution;* the solution given here is, however, my own and wholly different from that; hence I feel free here to make this known as my property since I leave the publication of the first solution, which was wholly trigonometric, to its discoverer himself."

[2] "Geometrische Construction des regelmässigen Siebenzehnecks," *Goettingische gelehrte Anzeigen*, Dec. 19, 1825, no. 203, p. 2025; Gauss, *Werke*, Band 2, pp. 186–187. See also *Bulletin des Sciences Mathématiques* ··· (Férussac), Vol. 5, 1826, pp. 299–300.

[3] This was reproduced in this MONTHLY, *1916*, 252.

[4] Unfortunately this proof has not been preserved.

While Gauss refers to two earlier synthetic constructions of Paucker,[1] as the only previous ones which he knew had been publicly discussed, he remarks that that of Erchinger is "different and carried through more in the spirit of pure geometry."

Stäckel reviews the discussions of the division of a circle into n equal parts, by means of imaginary quantities and the solution of the equation $x^n - 1 = 0$, by Cotes,[2] De Moivre,[3] Euler,[4] and Vandermonde,[5] and the development of their ideas by Gauss in connection with the problem of constructing a regular inscribed polygon.

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2843. Proposed by E. H. MOORE, University of Chicago.

Show that the maximum of the absolute value of

$$2(a + ib)(x + iy) + i(a + ib)(z + iw) + i(c + id)(x + iy),$$

where $i = \sqrt{-1}$ and a, b, c, d, x, y, z, w are real numbers for which

$$a^2 + b^2 + c^2 + d^2 = x^2 + y^2 + z^2 + w^2 = 1,$$

is $1 + \sqrt{2}$. Study the locus of point-pairs $P = (a, b, c, d)$, $Q = (x, y, z, w)$ of the unit-sphere in real four-space for which this absolute value assumes its maximum value.

2844. Proposed by J. L. RILEY, Stephenville, Texas.

Decompose into simple fractions the number $\frac{6\,0\,0\,2\,2\,8\,8\,0\,3\,5\,1}{1\,2\,7\,1\,6\,8\,0\,8\,7\,2\,0}$ (Gauss, *Disq. Arith.*, *Werke*, vol. 1, pp. 386–387).

2845. Proposed by E. L. POST, Princeton University.

Prove that if y_x is a solution of the functional equation

$$y_x = \frac{y_{x+1}^2}{x} + y_{x+1}$$

for positive integral values of x with $y_x > 0$, then

$$\lim_{x \to \infty} y_x \log x = 1.$$

2846.

Find the entire volume within the surface $x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$. (W. A. Granville, *Elements of Differential and Integral Calculus*, revised ed., 1911, p. 420.)

This equation, rationalized, is the equation of Steiner's quartic surface, every tangent plane to which cuts it in two conics. (Cf. Salmon-Rogers, *Analytic Geometry of Three Dimensions*, 5th ed., vol. 2, 1915, pp. 171, 201, 207, 213f. Also C. M. Jessop, *Quartic Surfaces* 1916, Chapter 7.)

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[1] (1) "Geometrische Verzeichnung des regelmässigen 17-Ecks und 257-Ecks in d. Kreis," *Jahresverhandl. d. kurländischen Gesellschaft für Litteratur und Kunst*, Mitau, Band 2, 1822. (2) *Die ebene Geometrie der geraden Linie und des Kreises.* Königsberg, 1823, p. 187. Paucker is also the author of: (3) *De divisione geometrica peripheriae circuli in XVII partes æquales*, Königsberg, 1817.

[2] R. Cotes, *Harmonia mensurarum, sive analysis et synthesis per rationum et angulorum mensuras promota.* Cambridge, 1722.

[3] A. De Moivre, *Miscellanea analytica*, London, 1730.

[4] L. Euler, *Introductio in analysin*, Lausanne, 1748, especially t. 1, cap 8: De quantitatibus transcendentibus ex circulo ortis.

[5] C. A. Vandermonde, "Remarques sur les problèmes de situation," *Histoire de l'Acad.*, année 1771, Paris 1774; *Mémoires*, p. 566. "Sur la résolution des équations," p. 365.

2847. Proposed by B. F. FINKEL, Drury College.

Convert $+\sqrt{R^2-x^2}$ into a Fourier series.

2848. Proposed by the late L. G. WELD.

A particle is attracted by a finite, uniform, material right line. Define its path, considering: (*a*) that the path is the envelope of the instantaneous lines of resultant attraction, as when the particle moves in a highly viscous medium (*i.e.*, without inertia); (*b*) that the particle moves freely (with inertia).

2849. Proposed by S. A. COREY, Des Moines, Iowa.

In the *Annals of Mathematics*, for April, 1911, Professor Byerly has given a method of approximately representing $f(x)$ in terms of $F_1(x)$, $F_2(x)$, $\cdots$, $F_r(x)$, in an interval $x_0 \leqq x \leqq x_1$. If $F_n(x) = x^n$ and if the mth derivative of $f(x)$ is zero when $x = 0$ for $m > r$, it is evident that Byerly's development becomes exact, and, therefore, identical with Maclaurin's development in the interval $0 \leqq x \leqq x_1$, provided $f(0) = 0$. If $f(0) \neq 0$, it is necessary to replace $f(x)$ by $f(x) - f(0)$ in Byerly's development. The coefficient of x^n in this development is D_n/D, where the D_n and D are certain determinants with elements of the type

$$A_n = \frac{x_1^{2n+1}}{2n+1}, \qquad B_{s,t} = \frac{x_1^{s+t+1}}{s+t+1}, \qquad C_n = \int_0^{x_1} f(x)x^n dx.$$

Prove that as r becomes infinite, D_n/D approaches $f^{[n]}(0)/n!$, the corresponding coefficient in the Maclaurin development, whenever $f(x)$ is analytic, and that any value of x within the range of convergence of the Maclaurin development may be substituted for x_1 without altering the value of D_n/D.

SOLUTIONS OF PROBLEMS.

416 (Algebra) [1914, 156; 1919, 312] Proposed by C. E. FLANAGAN, Wheeling, W. Va.

The sides of a given rectangle are a and b in which a rectangle is to be inscribed one of whose sides is c. Find the other side, using Euler's rule for quartics.

I. Solution and Discussion by Otto Dunkel, Washington University.

Let the given rectangle be $RSS'R'$ such that $RS = R'S' = a$ and $RR' = SS' = b$, and let $PP'Q'Q$ be a rectangle inscribed in it such that $QP = Q'P' = c$ and $PP' = QQ' = x$. Suppose further that the vertex P lies on RS, the vertex P' on SS' and so on in order. Let $SP = m$ and $SP' = n$; then from the four similar triangles in the figure, pairs of which are equal, we have at once

$$\frac{n}{x} = \frac{a-m}{c}, \qquad \frac{m}{x} = \frac{b-n}{c}, \qquad n^2 + m^2 = x^2, \tag{1}$$

and from the first two equations follow

$$n + m = \frac{(a+b)x}{c+x}, \qquad n - m = \frac{(a-b)x}{c-x}, \tag{2}$$

$$n(c^2 - x^2) = (ac - bx)x, \qquad m(c^2 - x^2) = (bc - ax)x.$$

Squaring and adding the equations in the first line of (2) and using the third equation in (1) we obtain

$$2 - \left(\frac{a+b}{c+x}\right)^2 - \left(\frac{a-b}{c-x}\right)^2 = 0. \tag{3}$$

It will be seen that the form of the equation above is convenient for separating the roots. On clearing of fractions it becomes

$$x^4 - (a^2 + b^2 + 2c^2)x^2 + 4abcx - c^2(a^2 + b^2 - c^2) = 0. \tag{3'}$$

The other side x of the inscribed rectangle must satisfy this equation, but in order to say that a given root of this equation determines the remaining side of an *inscribed* rectangle it must be shown that this root gives suitable values of m and n for such a rectangle.

From either (1) or (2) it follows that, if $x = c$, $a = b$. The case $a = b$ is easily disposed of, and the following results may be verified by the above equations. If $x \neq c$, $m = n$ and roots are $x_1 = \sqrt{2}\cdot a - c, x_2 = -(\sqrt{2}\cdot a + c)$. If $c < \sqrt{2}\cdot a$, x_1 gives the other side of an inscribed rectangle, which becomes a square if $c = a/\sqrt{2}$ and hence $m = n = a/2$. If $c > \sqrt{2}\cdot a$, then m, n and x_1 are negative and there is, of course, no inscribed rectangle, but there is a rectangle satisfying the analytical conditions, P and P' lying, respectively, on the prolongations of RS and of $S'S$. Also x_2 gives the side of a rectangle with P and P' on the prolongations of SR and SS'. For the double root of (3′), $x = c$, it follows that $n + m = a$. For each value of c such that $a \geqq c \geqq a/\sqrt{2}$, neither m nor n is negative and there is an inscribed square. If $c > a$, m and n have opposite signs and the square is not inscribed but the vertices P and P' are on the prolongations of SR and of $S'S$, respectively. The last two figures are the familiar figures in the proofs of the Pythagorean Theorem.

It will now be assumed that $a > b > 0$. If $c^2 \geqq a^2 + b^2$ it will be obvious that there are no inscribed rectangles. The roots of the equation, which are all real, are easily separated by the method given below for the case $c > a$. It is also easy to determine the positions of the four rectangles. In what follows it will be assumed that $c^2 < a^2 + b^2$.

The equation (3′) may be solved by any of the usual methods and there is no difficulty in writing out the expression for the roots other than that of the length and complication of the final result, and besides the result would be neither interesting nor useful. Important functions of the coefficients will be given and by the aid of these the roots may be obtained. Using the notation in Burnside and Panton's *Theory of Equations*, Vol. 1, (1904), page 121, we find for the reducing cubic of (3′)

$$4\theta^3 - I\theta + J = 0 \tag{4}$$

the following values for the quantities involved

$$H = -\frac{1}{6}(a^2 + b^2 + 2c^2), \qquad I = \frac{1}{12}(a^2 + b^2 - 4c^2)^2,$$
$$J = \frac{1}{216}[(a^2 + b^2 - 4c^2)^3 + 54c^2(a^2 - b^2)^2]. \tag{5}$$

The substitution $\theta = t + H$ gives Euler's cubic. The derivative of the left side of (4) vanishes for $\theta = \pm (a^2 + b^2 - 4c^2)/12$ and for these values of θ the left side takes on the extreme values $c^2(a^2 - b^2)^2/4$ and $[(a^2 + b^2 - 4c^2)^3 + 27c^2(a^2 - b^2)^2]/108$ and the product of these values gives $-\Delta/27$, where Δ is the discriminant (it may also be computed from $\Delta = I^3 - 27J^2$). Thus we have

$$\Delta = -\frac{1}{16}c^2(a^2 - b^2)^2[(a^2 + b^2 - 4c^2)^3 + 27c^2(a^2 - b^2)^2]. \tag{6}$$

It is now necessary to examine the last factor in Δ. Considering c^2 as the independent variable, it is seen that this factor has the maximum and minimum values $27a^2(a^2 - b^2)^2/2$ and $27b^2(a^2 - b^2)^2/2$ and hence vanishes only once for real values, passing from positive to negative values. Calling this root c_0^2 it is found to have the value

$$c_0^2 = \frac{a^2 + b^2}{4} + \tfrac{3}{8}(a^2 - b^2)^{2/3}[(a + b)^{2/3} + (a - b)^{2/3}] = \tfrac{1}{8}[(a + b)^{2/3} + (a - b)^{2/3}]^3. \tag{7}$$

If c^2 is greater than c_0^2 the roots of (3′) are all real or all imaginary, but since the equation has always one positive and one negative root the roots must be all real in this case. If c^2 is less than c_0^2 there are two imaginary roots in addition to the positive and negative root. For the value c_0^2 there are two equal roots.

In order to determine the number of inscribed rectangles for a given c several cases will be considered, the simplest of which seems to be the one for which $c > a$. This case may be treated without the use of the discriminant Δ. Setting $f(x)$ for the left side of (3), we have

$$f(0) = 2\left[1 - \frac{a^2 + b^2}{c^2}\right] < 0, \qquad f(b) = 2 - \left(\frac{a + b}{c + b}\right)^2 - \left(\frac{a - b}{c - b}\right)^2 > 0,$$
$$f\left(\frac{bc}{a}\right) = 2\left[1 - \frac{a^2}{c^2}\right] > 0, \qquad f(c) = -\infty, \qquad f\left(\frac{ac}{b}\right) = 2\left[1 - \frac{b^2}{c^2}\right] > 0. \tag{8}$$

The inequalities above follow from $c > a > b$ and $a^2 + b^2 > c^2$. Hence the positive roots lie in the intervals $(0, b)$, $(bc/a, c)$, $(c, ac/b)$. Since $f(-c) = -\infty$ there is a negative root smaller than $-c$. Since cx is the area of the rectangle determined, it can be inscribed only in the case of the root in the interval $(0, b)$. Taking m and n as two quantities determined from (2) by this root, we see from (3) that the equation $n^2 + m^2 = x^2$ is satisfied, and, since x is less than both bc/a and ac/b, both m and n are positive as shown by (2). The equations (1) result from (2) and show that m is less than a and that n is less than b. The equations (1) show also that c is the hypotenuse of a right triangle with the sides $a - m$ and $b - n$ and that c is perpendicular to x. This root furnishes then an inscribed rectangle. In the same way it will be seen that the second positive root makes n positive and m negative, also $n > b$ and $-m < n - b$, $-m > n - a$. The third root makes m positive and greater than a, and n negative, also $-n < m - b$, $-n > m - a$. In the figure for the second positive root Q lies on $R'R$ produced, P on RS produced, and P' on SS' produced. Similarly a figure can be drawn for the third positive root. In the case of the negative root m and n are positive and greater than a and b, respectively. Here P and P' lie, respectively, on the prolongations of SR and of SS', Q' and Q on the prolongations of $R'S'$ and of $R'R$.

If $c = a$ one root is b, and it will be found that of the two remaining positive roots one is less than b and gives an inscribed rectangle while the other is greater than a and hence cannot give an inscribed rectangle. These results show that a is greater than c_0, a fact which is easily verified directly.

The remaining cases may first be separated into two groups; I, $b < c < a$; and II, $0 < c \leqq b$. Again representing the left side of (3) by $f(x)$, we have

$$f'(x) = 2\left[\frac{(a+b)^2}{(c+x)^3} - \frac{(a-b)^2}{(c-x)^3}\right], \qquad f''(x) = -6\left[\frac{(a+b)^2}{(c+x)^4} + \frac{(a-b)^2}{(c-x)^4}\right]$$

(9)

$$f'\left(\frac{bc}{a}\right) = -\frac{4a^3b}{c^3(a^2-b^2)}, \qquad f\left(\frac{ab}{c}\right) = 2\,\frac{c^4 + a^2b^2}{(c^4 - a^2b^2)^2}(c^2 - a^2)(c^2 - b^2).$$

It is seen from (8) that $f(bc/a)$ is negative in both cases, and the value of the first derivative given in (9) shows that $f(x)$ is decreasing at this point, while the form of the second derivative shows that it continues to decrease as long as x is less than c. Hence, if there are positive roots less than c they must be less than bc/a, and the same reasoning used above will show that each such root gives an inscribed rectangle. When x is greater than c, (9) shows that $f'(x)$ is always positive, and hence $f(x)$ increases from $-\infty$ at $x = c$ and crosses the x-axis only once. From (9) it is seen that in case I the crossing point is beyond ab/c, and hence in this case there is no inscribed rectangle for the root. In case II, (9) shows that the root is less than or equal to ab/c. From (8) it will be seen that this root is greater than or equal to ac/b, since $f(ac/b)$ is negative or zero. It follows that it is also greater than bc/a, and the equations (2) will show that it gives an inscribed rectangle. This disposes of the root greater than c. The examination of the other two roots requires a consideration of the ratio of the sides a and b. An examination of (6) shows that; 1, if $a^2/b^2 \leqq 6\sqrt{3} - 9$, c_0 will lie in II; 2, if $a^2/b^2 > 6\sqrt{3} - 9$, c_0 will lie in I. Hence for case II, we have $c > c_0$ and therefore two more positive roots. These give inscribed rectangles and they are the only two in this case. For I2, we have two inscribed rectangles if $c > c_0$; one, if $c = c_0$; none if $c < c_0$. Passing now to III it follows that if $c > c_0$ there are in all three inscribed rectangles; if $c = c_0$, two; if $c < c_0$, only one. Finally for II2 there is no other inscribed rectangle than the one mentioned above. The roots not giving inscribed rectangles may be interpreted in the same way as in the first part of this discussion.

The character of the roots was obtained at the beginning of this discussion by an examination of the discriminant of the quartic (3') since the wording of the proposed problem seemed to indicate that a discussion of this equation was desirable with reference to Euler's solution of the quartic. But it will be found that that procedure is very tedious as compared with the study of (3), and besides the study of this latter equation does not require any knowledge of the theory of equations. Thus from (9) it will be seen that $f'(x)$ vanishes only once for real values of x and hence there will be two more real roots (in addition to the two always present) or not according as the value of $f(x)$ at this vanishing point is positive or negative. The value of this maximum is easily found, but it is well to find first the value of c_0 for the double root. This is easily found by eliminating x from $f(x) = 0$ and $f'(x) = 0$ and it follows that

$$c_0 = \frac{1}{2\sqrt{2}}\left[(a+b)^{2/3} + (a-b)^{2/3}\right]^{3/2}, \tag{10}$$

which is easily verified to agree with the previous result. The corresponding double root is then found to be

$$(11) \qquad x_0 = \frac{c_0^{1/3}}{2}[(a+b)^{2/3} - (a-b)^{2/3}] = \sqrt{\frac{a^2+b^2-c_0^2}{3}} = \frac{6abc_0}{a^2+b^2+8c_0^2}.$$

The remaining two roots are obtained by adding to the negative of the double root

$$(12) \qquad \pm\sqrt{\frac{a^2+b^2+8c_0^2}{3}}.$$

The maximum value of $f(x)$ at $x = (x_0/c_0)c$ will be found to be

$$(13) \qquad 2\left[1 - \frac{c_0^2}{c^2}\right],$$

and it gives the same results as the discriminant and in a much more evident form. This shows that the first two positive roots are separated by $(x_0/c_0)c$.

In Osgood's *Calculus*, page 404, is given a solution of the case $a = 15$, $b = 10$, $c = 1$. Here $c_0 = 13.741$, $x_0 = 6.738$, and since $a^2/b^2 = 2.25 > 6\sqrt{3} - 9$ this example falls in II2 and there is only one inscribed rectangle. If $c = 13$ there is no inscribed rectangle. If $c = 14$ there are two inscribed rectangles, one having the side 5.+ and the other the side 8.+.

II. Note on the Preceding by H. P. Manning, Brown University.

There is no particular distinction between the sides c and x of the inscribed rectangle. We can call them x and y, and equation (3) or (3′) will then be represented by a curve of the fourth degree in which many of the results of the above discussion appear graphically.

Moreover, m and n satisfy the equation $m^2 - n^2 = am - bn$, and so are represented by the points of an equilateral hyperbola, which passes through the vertices of the given rectangle if we lay off m and n from its lower left-hand corner.

273 (Number Theory) [1917, 427]. Proposed by V. M. SPUNAR, Chicago, Ill.

The ratio of the chances that all numbers ending in 1 or 9 and those ending in 3 or 7 are composite is $3 : 2$[1].

Note by Norman Anning, University of Maine.

Since

$$1 \times 1 \equiv 3 \times 7 \equiv 9 \times 9 \equiv 1 \pmod{10}, \qquad 1 \times 3 \equiv 7 \times 9 \equiv 3 \pmod{10},$$

$$1 \times 7 \equiv 3 \times 9 \equiv 7 \pmod{10}, \qquad 1 \times 9 \equiv 3 \times 3 \equiv 7 \times 7 \equiv 9 \pmod{10},$$

the conclusion might be drawn that in the long run as many primes end in 1 as in 9 and as many end in 3 as in 7 and that there would be more of the latter than of the former. A census of primes taken over a considerable range supports these statements but does not point towards the ratio "3 : 2." Counting cases up to 3,200, a number chosen at random, shows the following results:

110 primes end in 1, 113 in 3, 116 in 7 and 111 in 9.

Since these numbers are so nearly in equilibrium and since primes are so perfectly lawless no statement could be hazarded about the distribution in a larger interval.

2728 [1918, 397]. Proposed by NORMAN ANNING, University of Maine.

A material triangle of uniform density and thickness is of such a shape that when suspended from the vertices in succession, the lower sides have slopes of $1 : 1$, $1\frac{1}{2} : 1$, and $3 : 1$. Construct the triangle given that the shortest side is 10 inches.

By definition, an $a : 1$ slope makes an angle with the vertical whose tangent is a.

[1] The enunciation of the problem is not clear. The chance that all numbers ending in 1 or 9 are composite numbers is zero. — EDITORS.

Solution by the Proposer.

As this triangle is designed to show the slope of earth embankments the word 'slope' is defined as it is understood by the civil engineer: the ratio of horizontal to vertical distance.

Let AD, BE, CF be the medians of the required triangle, ABC, then

$$2\cot ADB = \cot C - \cot B, \tag{1}$$

$$2\cot BEC = \cot A - \cot C, \tag{2}$$

$$2\cot CFA = \cot B - \cot A, \tag{3}$$

and hence $\cot ADB + \cot BEC + \cot CFA = 0$. If we take $\cot ADB = \frac{2}{3}$ then from the assigned values: (a) $\cot BEC = -1$, $\cot CFA = \frac{1}{3}$; (b) $\cot BEC = \frac{1}{3}$, $\cot CFA = -1$. Considering case (a), these values inserted in (1), (2), (3) give two independent equations, which are to be combined with the identity,

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1. \tag{4}$$

Putting, from (2) and (3), $\cot B = \cot A + \frac{2}{3}$, $\cot C = \cot A + 2$, equation (4) becomes

$$9\cot^2 A + 16\cot A + 1 = 0;$$

whence,

$$\cot A = \frac{-8 \pm \sqrt{55}}{9}; \quad \cot B = \frac{-2 \pm \sqrt{55}}{9}; \quad \text{and} \quad \cot C = \frac{10 \pm \sqrt{55}}{9}.$$

Since a triangle can have only one obtuse angle only one cotangent can be negative. Consequently, the $(-)$ before the radical does not lead to a solution. Using the $(+)$ sign,

$$A = 93^\circ\,42'\,42'', \qquad B = 58^\circ\,57'\,37'', \qquad C = 27^\circ\,19'\,41''.$$

When $c = 10.00$, $a = 21.74$ and $b = 18.66$.

Case (b) can be treated in the same way and we get

$$\cot A = \frac{8 + \sqrt{55}}{9}, \qquad \cot B = \frac{-10 + \sqrt{55}}{9}, \qquad \cot C = \frac{2 + \sqrt{55}}{9}.$$

2745 [1919, 37]. Proposed by G. I. HOPKINS, Manchester, N. H.

A recent English publication contains the following method of constructing a regular polygon of 17 sides: Draw the radius CB perpendicular to the diameter AQ of the circle whose center is B. On BC lay off BD equal to one-fourth of BC. On BA, lay off BE and draw DE making angle BDE one fourth of angle BDA. On BQ lay off BF and draw DF, making angle FDE 45°. On AF as diameter, construct semi-circle FHA intersecting CB in H. With E as center and EH as radius construct semi-circle LHK intersecting CB in H. At the points L and K draw the ordinates NL and MK. Bisect the arc NM and let P be the point of bisection. Then the chord $NP(=MP)$ is a side of the regular polygon of 17 sides. Is the method of construction correct?

I. Solution by C. H. Chepmell, Hove, England.

The abscissas BK, BL, and their ordinates to M and N, make the angles MBA, NBA equal to $10\pi/17$, and $6\pi/17$ respectively. Consequently the difference of these two angles, the angle MBN, is equal to $4\pi/17$. This justifies the claim of the construction.

But tables of all the trigonometrical functions of $n\pi/17$ are not readily available; and it may be more satisfactory if we outline the connection between the numerical value of NM and the numerical value of some function given in a standard work.

Taking the radius of the circle ACQ as unity, we find

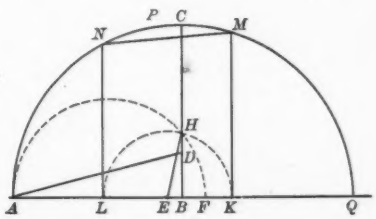
$$BE = \frac{-1 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}}}{16},$$

$$BF = \frac{+1 - \sqrt{17} + \sqrt{34 - 2\sqrt{17}}}{16},$$

$$EH = \frac{\sqrt{2(\sqrt{17} - 3)(2\sqrt{17} + \sqrt{34 + 2\sqrt{17}})}}{16} = \frac{\beta}{16},$$

$$AK = \frac{+17 + \sqrt{17} - \sqrt{34 + 2\sqrt{17}} + \beta}{16},$$

$$AL = \frac{+17 + \sqrt{17} - \sqrt{34 + 2\sqrt{17}} - \beta}{16},$$



$$KQ = \frac{+15 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}} - \beta}{16},$$

$$LQ = \frac{+15 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}} + \beta}{16},$$

$$LN = \sqrt{AL \cdot LQ} = \tfrac{1}{8}\sqrt{+34 + 2\sqrt{17} + 2\sqrt{34 + 2\sqrt{17}} - 2\beta'},$$

$$KM = \sqrt{AK \cdot KQ} = \tfrac{1}{8}\sqrt{+34 + 2\sqrt{17} + 2\sqrt{34 + 2\sqrt{17}} + 2\beta'},$$

$$[\beta' = \sqrt{2(\sqrt{17} - 3)(2\sqrt{17} - \sqrt{34 + 2\sqrt{17}})}; \quad \text{and} \quad \beta \times (-1 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}}) = 4 \cdot \beta'],$$

$$LN \times KM = \tfrac{1}{64}\sqrt{2(\sqrt{17} + 3)(2\sqrt{17} + \sqrt{34 - 2\sqrt{17}})} = \frac{4\alpha}{64},$$

$$NM^2 = LK^2 + (KM - LN)^2$$

$$= 4 \cdot EH^2 + KM^2 + LN^2 - \frac{8\alpha}{64}$$

$$= \frac{+136 - 8\sqrt{17} + 8\sqrt{34 - 2\sqrt{17}} - 8\alpha}{64},$$

$$[\alpha' = \sqrt{2(\sqrt{17} + 3)(2\sqrt{17} - \sqrt{34 - 2\sqrt{17}})}; \quad \text{and} \quad 4\alpha = \alpha' \times (-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}})],$$

$$\therefore \quad NM^2 = \frac{+136 - 8\sqrt{17} + 8\sqrt{34 - 2\sqrt{17}} - 2(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}})\alpha'}{64}.$$

And, in the circle with radius equal to unity, NM represents the value $2 \sin (2\pi/17) \times 1$; and therefore

$$4 \cos^2 \frac{2\pi}{17} = 4 - NM^2$$

$$= \frac{+120 + 8\sqrt{17} - 8\sqrt{34 - 2\sqrt{17}} + 2(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}})\alpha'}{64}$$

$$= \left[\frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + \alpha'}{8}\right]^2$$

and

$$2 \cos \frac{2\pi}{17} = \frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + \sqrt{2(\sqrt{17} + 3)(2\sqrt{17} - \sqrt{34 - 2\sqrt{17}})}}{8}.$$

And this value of $2 \cos (2\pi/17)$ will be found to agree with that given in Klein's *Famous Problems in Elementary Geometry* (Beman & Smith), though our α' is there written in a different form.

II. Historical Note by R. C. Archibald, Brown University.

This method of construction is due to H. W. Richmond, *Quarterly Journal of Mathematics*, Volume 26, 1893, pp. 206–207; and *Mathematische Annalen*, Volume 67, 1909, pp. 460–461. It is reproduced on page 34 of H. P. Hudson's *Ruler and Compasses*, London, 1916.

Various constructions of the regular polygon of seventeen sides were reviewed by R. Goldenring in his *Die elementargeometrischen Konstruktionen des regelmässigen Siebzehnecks* (Leipzig, 1915), but many omissions in this professedly complete survey were noted by the writer in the *Bulletin of the American Mathematical Society*, vol. 22, 239-246. The first solution in an English publication, given by Lowry in 1819,[1] was reproduced in this *Monthly*[2] in 1899 and 1914. Other solutions and historical notes are set forth in the articles printed above, pages 322–326.

2767 [1919, 171]. Proposed by W. W. Johnson, U. S. Naval Academy.

Let the complex quantities p, q, and r satisfy the relation $p^2 + q^2 + r^2 = 0$; prove that the corresponding vectors OP, OQ, and OR are such that if any two of them are taken as conjugate semi-diameters of an ellipse, the third lies on the minor axis, and its length is the distance from the center to either focus.

Solution by A. Pelletier, Montreal, Can.

Let $(x^2/a^2) + (y^2/b^2) = 1$, be the equation of the ellipse having OP and OQ for conjugate semi-diameters ($2a$ and $2b$ being the axes, and $a \geqq b$). If α, α', α'' are the respective arguments of

[1] *The Mathematical Repository*, new series, vol. 4, p. 160; Lowry's proof occupies pages 160-168.
[2] Volume 6, p. 239 and volume 21, p. 252.

OP, OQ, and OR, we have

$$OP^2(\cos 2\alpha + i \sin 2\alpha) + OQ^2(\cos 2\alpha' + i \sin 2\alpha') = -OR^2(\cos 2\alpha'' + i \sin 2\alpha''),$$

from datum; hence,

$$OP^2 \cos 2\alpha + OQ^2 \cos 2\alpha' = -OR^2 \cos 2\alpha'' \tag{1}$$

and

$$OP^2 \sin 2\alpha + OQ^2 \sin 2\alpha' = -OR^2 \sin 2\alpha''. \tag{2}$$

Now P and Q being points on the ellipse, we have from known properties,

$$OP^2 \cos^2 \alpha + OQ^2 \cos^2 \alpha' = a^2, \qquad OP^2 \sin^2 \alpha + OQ^2 \sin^2 \alpha' = b^2;$$

hence, $OP^2 \cos 2\alpha + OQ^2 \cos 2\alpha' = a^2 - b^2$, and (1) becomes

$$-OR^2 \cos 2\alpha'' = a^2 - b^2. \tag{3}$$

Also, from known properties concerning the ends of conjugate diameters,

$$OP^2 \sin 2\alpha = -OQ^2 \sin 2\alpha';$$

hence, (2) becomes

$$-OR^2 \sin 2\alpha'' = 0. \tag{4}$$

It follows from (3) and (4), that $2\alpha'' = 180°$ or $540°$, and $OR^2 = a^2 - b^2$, that is, $OR = \sqrt{a^2 - b^2}$, the distance from the center to focus, and $\alpha'' = 90°$ or $270°$, which shows that OR lies on the minor axis.

Also solved by H. HALPERIN, A. M. HARDING, and H. L. OLSON.

2780 [1919, 311]. Proposed by ELMER LATSHAW, West Philadelphia, Pa.

A quadrilateral whose sides are a, $2a$, $3a$, $4a$ is inscribed in a circle. Find the radius of the circle.

I. SOLUTION BY H. S. UHLER, Yale University.

The interest in this problem may be enhanced by giving a perfectly general solution. Let the sides of any convex inscriptible quadrilateral be denoted by a_1, a_2, a_3, a_4. A diagonal c may be drawn dividing the quadrilateral into two non-overlapping triangles the sides of which are a_1, a_2, c and a_3, a_4, c, respectively. If the angle between a_1 and a_2 be symbolized by C, the angle between a_3 and a_4 must be $180° - C$. Accordingly

$$c^2 = a_1^2 + a_2^2 - 2a_1a_2 \cos C,$$

$$c^2 = a_3^2 + a_4^2 + 2a_3a_4 \cos C.$$

Eliminating $2\cos C$ we find

$$c^2 = \frac{(a_1a_3 + a_2a_4)(a_2a_3 + a_4a_1)}{a_1a_2 + a_3a_4}. \tag{1}$$

The area of a plane triangle having the sides a_1, a_2, c is given by either member of the following equation

$$\frac{a_1a_2c}{4R} = \sqrt{s(s - a_1)(s - a_2)(s - c)}, \tag{2}$$

where $2s = a_1 + a_2 + c$, and R denotes the radius of the circumscribed circle.

Substituting the trinomial value of s in equation (2) we obtain

$$\frac{a_1a_2c}{R} = \sqrt{[(a_1 + a_2)^2 - c^2][c^2 - (a_1 - a_2)^2]}. \tag{3}$$

Replacing c in equation (3) by expression (1) we eventually find that

$$R = \frac{\sqrt{(a_1a_2 + a_3a_4)(a_1a_3 + a_4a_2)(a_2a_3 + a_4a_1)}}{\sqrt{(a_2 + a_3 + a_4 - a_1)(a_3 + a_4 + a_1 - a_2)(a_4 + a_1 + a_2 - a_3)(a_1 + a_2 + a_3 - a_4)}}, \tag{4}$$

or

$$R = \frac{1}{4K}\sqrt{(a_1a_2 + a_3a_4)(a_1a_3 + a_4a_2)(a_2a_3 + a_4a_1)}. \tag{5}$$

where if $2S = a_1 + a_2 + a_3 + a_4$, $K = \sqrt{(S - a_1)(S - a_2)(S - a_3)(S - a_4)}$ = area of quadrilateral.

The denominator of formula (4) brings out the geometrically-evident fact that each side of the quadrilateral must not exceed the sum of the remaining three sides. When the quadrilateral degenerates into a straight line, formulas (4) and (5) give $R = \infty$, as they should. These formulæ also show explicitly that the order or succession of the sides has no effect on the value of R, a fact which is obvious geometrically since the sum of the arcs subtended by the four sides of the quadrilateral equals the entire circumference.

The answer to the given problem may be obtained at once from formula (5) by substituting $a, 2a, 3a, 4a, 5a$ for a_1, a_2, a_3, a_4, S respectively. It is

$$R = \frac{a\sqrt{385}}{4\sqrt{6}} = (2\cdot 002602\cdots)a.$$

II. Solution by Bing Chin Wong, Berkeley, Calif.

Let $ABCD$ be the polygon with sides $AB = a$, $BC = 2a$, $CD = 3a$, $DA = 4a$ inscribed in the circle with O as center. Join O to A, B, C, D. Then

$$\angle AOB + \angle BOC + \angle COD + \angle DOA = 2\alpha + 2\beta + 2\gamma + 2\delta = 2\pi,$$

or

$$\alpha + \beta + \gamma + \delta = \pi.$$

Then

$$\cos(\alpha + \beta) + \cos(\gamma + \delta) = 0,$$

or

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\gamma\cos\delta - \sin\gamma\sin\delta = 0. \qquad (I)$$

Let r be the radius of the circle. We obtain from the figure,

$$\sin\alpha = \frac{a}{2r}, \quad \sin\beta = \frac{a}{r}, \quad \sin\gamma = \frac{3a}{2r}, \quad \sin\delta = \frac{2a}{r};$$

and

$$\cos\alpha = \frac{\sqrt{4r^2 - a^2}}{2r}, \quad \cos\beta = \frac{\sqrt{r^2 - a^2}}{r}, \quad \cos\gamma = \frac{\sqrt{4r^2 - 9a^2}}{2r}, \quad \cos\delta = \frac{\sqrt{r^2 - 4a^2}}{r}.$$

Substituting these values in (I) and multiplying by $2r^2$, we have

$$\sqrt{4r^2 - a^2}\sqrt{r^2 - a^2} + \sqrt{4r^2 - 9a^2}\sqrt{r^2 - 4a^2} = 7a^2.$$

Squaring, collecting terms, and dividing by 2, we have

$$\sqrt{(4r^2 - a^2)(r^2 - a^2)(r^2 - 4a^2)(4r^2 - 9a^2)} = 6a^4 + 15a^2r^2 - 4r^4.$$

Squaring again and collecting terms, we have

$$96a^4r^4 = 385a^6r^2, \quad \text{or} \quad r^2 = 385a^2/96,$$

and, therefore,

$$r = a\sqrt{385/96} = a\sqrt{2310/24} = 2.0026a.$$

Also solved by H. C. Bradley, H. N. Carleton, S. A. Corey, Laura Guggenbuhl, T. F. Noismann, H. L. Olson, A. Pelletier, J. B. Reynolds, and the Proposer.

2781 [1919, 311]. Proposed by J. L. Riley, Stephensville, Texas.

Show that the asymptotic lines on a pseudospherical surface are curves of constant torsion.[1]

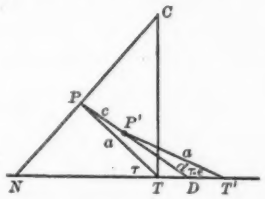
Solution by Otto Dunkel, Washington University.

The Theorem of Euneper states that the square of the torsion of an asymptotic line at any point of a surface is equal to the total curvature of the surface (with sign changed) at the point (Eisenhart, *Differential Geometry*, p. 140). Since it is *known* that the total curvature of a pseudospherical surface is constant, it follows at once that the torsion is constant.

[1] This problem is given as an example in Eisenhart's *Differential Geometry*, p. 290.—Editors.

The pseudosphere is obtained by rotating the tractrix about its asymptote. That its total curvature is constant may be proved as follows: In any surface of revolution the principal radii of curvature are the radius of curvature of the meridian section and the length of the normal at the given point (Goursat-Hedrick, vol. 1, p. 505). As is well known the length of the tangent for the tractrix is a constant a; in fact, this is usually given as its definition. Let P and P' be two neighboring points on a tractrix, $PT = P'T' = a$ the tangents at these points, and τ and $\tau - \epsilon$, their *acute* inclinations to the axis. Produce $PP' = c$ to cut the axis in D and set $P'D = d$; then

$$\frac{c+d}{d} = \frac{a \sin \tau}{a \sin (\tau - \epsilon)} \quad \text{or} \quad \frac{c}{d} = \frac{\sin \tau - \sin (\tau - \epsilon)}{\sin (\tau - \epsilon)}.$$



Thus

$$\frac{c}{\epsilon} = d \frac{\cos\left(\tau - \frac{\epsilon}{2}\right)}{\sin (\tau - \epsilon)} \frac{\sin \frac{\epsilon}{2}}{\frac{\epsilon}{2}}.$$

As ϵ approaches zero, d approaches a and c/ϵ approaches the radius of curvature R. Hence $R = a \cot \tau$, and this shows that the line joining the center of curvature C with T is perpendicular to the axis. If PN is the normal at P, the right triangle CTN gives $CP \times PN = a^2$. (See Goursat-Hedrick, vol. 1, p. 441 for an analytical proof.) Thus the total curvature at every point of the pseudosphere is $-1/a^2$ and the torsion of an asymptotic line is $1/a$.

Also solved by J. B. REYNOLDS.

2782 [1919, 311]. Proposed by WARREN WEAVER, University of Wisconsin.

A great number, n, of jackstraws are jumbled up in such a way that any one is as likely to have one direction as another. Show that the probable number that make an angle lying between θ_1 and θ_2 as measured from any given direction is equal to $\frac{n(\cos \theta_1 - \cos \theta_2)}{2}$.

SOLUTION BY H. L. OLSON, University of Wisconsin.

Let us imagine the jackstraws all to have been translated in space so as to have *corresponding* ends coincident, and let us consider a sphere of unit radius with its center at this common end-point. The jackstraws satisfying the specified condition will then intersect the sphere in the points of a zone whose bases are circles with angular radii θ_1 and θ_2, respectively. Since the area of this zone is $2\pi(\cos \theta_1 - \cos \theta_2)$ (if $\theta_1 < \theta_2$) and the area of the entire sphere is 4π, the probable number of jackstraws making angles between θ_1 and θ_2 with a given line is $\frac{n(\cos \theta_1 - \cos \theta_2)}{2}$.

2789 [1919, 414]. Proposed by KURT LAVES, University of Chicago.

Given a quadrilateral $ABCD$ for which $|AC - BC| > |AD - BD|$[1] $(AC + BC < AD + BD)$ to construct, by means of the ruler and compass only, the pair of tangents from D to the hyperbola (ellipse) for which A and B are the foci and C a point on the hyperbola (ellipse).

SOLUTION BY A. PELLETIER, Montreal, Can.

Ellipse: D is outside the curve and the solution is always possible. With B as a center and a radius of length $AC + BC$ describe the circle x; with D as a center and DA as a radius, describe the circle y. The circles x and y intersect at M and N. The perpendiculars erected on AM, AN, at the mid-points of these lines are the required tangents, as is well known.

Hyperbola: The construction is the same as above, the radius of x being $|AC - BC|$.

[1] This expression was inadvertently omitted when published originally.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

At Elmira College, Miss MARY C. SUFFA, instructor in mathematics at Beloit College, has been appointed head of the department of mathematics and Miss FRANCES W. WRIGHT, of Brown University, has been appointed instructor.

At Queen's University, Assistant Professor C. F. GUMMER has been promoted to be an associate professor, and Lecturer Dr. NORMAN MILLER to be an assistant professor. Mr. A. WOODS, a tutor in mathematics during 1919–20, has been appointed lecturer at Western University, London, Ontario.

Dr. J. W. CAMPBELL, associate professor of mathematics and astronomy at the University of Iowa, has been appointed assistant professor of mathematics at the University of Alberta.

Science states that Dr. J. P. MUSSELMAN, of Washington University, has been appointed associate in mathematics at the John Hopkins University. Dr. Musselman is the national president of the Gamma Alpha Graduate Scientific Fraternity.

Professor OSWALD VEBLEN has been elected the representative of mathematics in the Executive Committee of the Division of Physical Sciences of the National Research Council.

Professor IDA BARNEY, of Meredith College, has been appointed assistant professor of mathematics at Smith College.

At Oberlin College, Associate Professor W. D. CAIRNS has been promoted to be professor of mathematics and head of the department in place of Professor FREDERICK ANDEREGG who has retired after thirty-three years of service as a teacher.

Professor D. R. CURTISS has been appointed lecturer at Harvard University for the second semester of 1920–21. He will give courses in "The analytical theory of heat and problems in elastic vibration" and "Functions defined by linear differential equations of the second order."

Miss ZOE FERGUSON, of the Central High School and Junior College, St. Joseph, Mo., has accepted a position in the Crane Technical High School and Junior College of Chicago.

Dr. L. R. FORD, of Harvard University, has been appointed assistant professor of mathematics at the Rice Institute.

Professor T. R. HOLLCROFT was appointed in charge of the department of mathematics at Wells College a year ago in place of Miss ANNA L. VAN BENSCHOTEN who retired on account of ill health and is now living in California.

At the University of Pennsylvania, Dr. J. R. KLINE, associate at the University of Illinois, [*1919*, 322] has been appointed assistant professor of mathematics.

At the University of Texas, Assistant Professor R. L. MOORE, of the University of Pennsylvania, has been appointed associate professor of pure mathematics, and Dr. JESSIE M. JACOBS [*1920*, 92] instructor in pure mathematics.

Messrs. F. H. MURRAY and J. L. WALSH have been appointed Sheldon Fellows by Harvard University and expect to spend the coming year studying mathematics in Paris.

Miss GERTRUDE SMITH, of Vassar College, has been promoted to an assistant professorship of mathematics.

Assistant professor LOUISA M. WEBSTER, of Hunter College, has been promoted to an associate professorship. She is vice-president of the Association of Mathematics Teachers of the Middle States and Maryland.

At Cambridge University, England, Mr. J. E. LITTLEWOOD, of Trinity College, has been appointed Cayley lecturer in mathematics and Mr. J. H. GRACE, of Peterhouse, has been reappointed University lecturer in mathematics.

Dr. M. G. HUMBERT, professor of analysis at the Ecole Polytechnique for twenty-five years, has resigned and been appointed honorary professor.

Dr. RENÉ GARNIER, chargé d'un cours in rational and applied mechanics at the University of Poitiers, has been appointed professor of rational and applied mechanics in place of Professor FRÉCHET who had been transferred to the University of Strasbourg.

Professor E. P. J. VESSIOT, of the University of Paris, has been appointed sub-director of the Ecole Normale Supérieure in place of Professor EMILE BOREL [*1920*, 280] who resigned after ten years of service, and has been appointed honorary director. Louis Pasteur, L. E. Bertin, director of French naval construction, and Jules Tannery have occupied this post in earlier years.

Professor C. CARATHÉODORY, of the University of Berlin, has been appointed professor of mathematics at the national university in Athens. Professor Carathéodory's place at Berlin has been filled by the appointment of Professor L. E. BROUWER, of the University of Amsterdam.

Other appointments in Germany are as follows: Dr. R. COURANT, privatdozent at the University of Göttingen to be ordinary professor of mathematics at the University of Münster.—Professor H. HAPPEL, of the University of Tübingen, to be ordinary professor of applied mathematics at the Technische Hochschule, Breslau.—Dr. O. HAUPT, privatdozent at the Technische Hochschule, Karlsruhe, to be ordinary professor at the University of Rostock.—Professor H. JUNG, of the University of Kiel, to be ordinary professor of mathematics at the University of Halle.—Professor R. v. MISES, of the Technische Hochschule, Dresden, to be ordinary professor of applied mathematics at the University of Berlin.—Dr. R. H. WEBER, extraordinary professor of physics at the University of Rostock, to be honorary professor there.

Dr. P. VAN GEER, professor at the University of Leyden since 1867, died at The Hague, October 3, 1919, aged seventy-eight years. He was the author of various books on geometrical topics, from his Leyden dissertation on geodetic lines of an ellipsoid (1862), to his foundations of synthetic geometry (1900).

Dr. M. HAID, professor of practical geometry and higher geodesy at the Technische Hochscule, Carlsruhe, died November 5, 1919.

Dr. P. G. STACKEL, professor at the University of Heidelberg since 1912, died December 12, 1919, aged fifty-seven years. His section of the *Encyklopädie* on "Elementare Dynamik der Punktsysteme und starren Körper," and his activities in connection with the editing of the works of the Bolyais and Euler will be recalled.

SRINIVASA RAMANUJAN, brilliant Indian mathematician of "astonishing individuality and power" (Hardy), and with "powers as remarkable in their way as those of any living mathematician" (Hardy), died at Chetput, Madras Presidency on April 26, 1920 at the early age of thirty-one. He was a fellow of Trinity College, Cambridge, and a research fellow of the University of Madras, and was elected the first Indian fellow of the Royal Society, London, when only twenty-eight years of age. A portrait and biographical sketch from which quotation was made, have been already referred to in this MONTHLY, *1920*, 74. Quotation from a sketch by Professor G. H. Hardy is made elsewhere in this issue (page 316). The report of Professor Hardy to the University of Madras on "Mr. S. Ramanujan's mathematical work in England," which was published in *The Journal of the Indian Mathematical Society* for February, 1917, surveyed about a dozen papers written after his arrival in England in April, 1914. Some others are referred to in a memoir by Hardy and Ramanujan in the 1919 volume (95A) of the *Proceedings of the Royal Society of London*. It will be recalled that chapter 3 in the second volume of P. A. MacMahon's *Combinatory Analysis* (1916) is entitled "Ramanujan's identities."

Dr. C. A. LAISANT, examiner for admission to the Ecole Polytechnique since 1909, died May 6, 1920, in his seventy-eighth year. He was a nephew of the

celebrated ophthalmologist Ange Guépin of Nantes. From 1876 to 1893 he was a member of the chamber of deputies in the French parliament. He was the author of about fifteen volumes of mathematical works and of a large number of mathematical papers published, for the most part, in the *Bulletin de la Société Mathématique de France*, and in *Comptes Rendus de l'Association francaise pour l'Avancement des Sciences*. His book entitled *La Mathématique: Philosophie Enseignement* (Paris, 1898; second edition revised, 1907), was translated into Russian (St. Petersburg, sixth edition, 1912) and into English (London, 1913); and his six volume *Recueil de Problèmes de Mathématiques* (1893–96) is a very useful bibliographic aid. Details concerning his collaboration with Lemoine in founding *L'Intermédiaire des Mathématiciens*[1] have been set forth by Professor D. E. Smith in this MONTHLY, *1896*, 32. As to Dr. Laisant's non-mathematical publications and political life, reference may be made to his biography in G. Vapereau's *Dictionnaire Universel des Contemporains*, sixième édition, Paris, 1893.

The American Association for the advancement of Science made the following grants for mathematical research in 1919: Three hundred dollars to Professor SOLOMON LEFSCHETZ of Kansas University to assist in the publication of his memoir on algebraic surfaces, which was awarded the Bordin prize of the Paris Academy of Sciences [*1920*, 143], one hundred dollars to Dr. OLIVE C. HAZLETT of Mount Holyoke in support of her work on the theory of hypercomplex numbers and variants.

At the annual meeting of the American Academy of Arts and Sciences in May, Professor J. S. HADAMARD was elected a foreign honorary member in the division of mathematics and astronomy. [Cf. *1920*, 239.]

In the Division of Physical Sciences of the National Research Council, Professor G. D. BIRKHOFF has been appointed the new member at large and Professor E. R. HEDRICK the representative of the Mathematical Association of America.

The following are the members of the American Section of the International Mathematical Union [*1920*, 282] for 1920: Professors FRANK MORLEY, F. N. COLE, H. S. WHITE, OSWALD VEBLEN, L. E. DICKSON, VIRGIL SNYDER, G. A. MILLER, G. D. BIRKHOFF, E. B. VAN VLECK representing the American Mathematical Society; H. E. SLAUGHT, E. V. HUNTINGTON, and D. E. SMITH, representing the Mathematical Association of America, and E. R. HEDRICK as representing the Association in the Division of Physical Sciences; M. W. HASKELL representing the American Association for the Advancement of Science, LEIGH PAIGE representing the American Physical Society, and E. H. MOORE representing the National Academy.—At this writing the American Astronomical Society has appointed no representative.—This Section has appointed Professors L. E. DICK-

[1] Laisant's own account of this may be found in his memoir of Lemoine in *L'Enseignement Mathématique*, vol. 14, 1912, p. 182.

SON and L. P. EISENHART to be the representatives of American mathematicians at the meetings of the Union, which convene at Brussels about September 20. These representatives have been authorized to appoint alternates on arriving at Brussels. Other American mathematicians who expect to attend the Congress which opens at Strasbourg on September 22 are SOLOMON LEFSCHETZ, J. L. WALSH, F. H. MURRAY, NORBERT WIENER, J. S. TAYLOR, THOMAS BUCK, and D. E. SMITH.

Reference has been made in earlier numbers of this MONTHLY [*1916*, 270–271; *1919*, 372], to generous provisions made by MITTAG-LEFFLER and his wife for the founding of a Mathematical Institute in Stockholm. It is now announced that Cornell University has received an anonymous gift from a professor and his wife of a trust fund for an Institute of pure and applied mathematics. The gift amounts to $50,000 and is to be held in trust for a hundred years and allowed to accumulate. At five per cent. interest compounded annually this accumulation would amount to about six and one half million dollars; at six per cent. to over three times this amount.

On the recommendation of the National Academy of Sciences the Barnard medal for meritorious service to science has been conferred by Columbia University on Professor ALBERT EINSTEIN, of Berlin, in recognition "of his highly original and fruitful development of the fundamental concepts of physics through application of mathematics."

The adjudicators of the Hopkins prize of the Cambridge Philosophical Society have made the following awards: for the period 1903–6 Dr. W. BURNSIDE, of Pembroke College, for investigations in mathematical science; for the period 1906–9 to Professor G. H. BRYAN, of Peterhouse, for investigations in mathematical physics, including aerodynamic stability; and for the period 1909–12 to Mr. T. R. WILSON, of Sidney Sussex College, for investigations in physics, including the paths of radio-active particles.

In March, 1918, mathematicians from Athens, Piraeus, and the Provinces, gathered at Athens and organized a Greek Mathematical Society. In 1918 there were 93 members and the following officers: Honorary Presidents, J. HATZIDAKIS and D. AIGINITIS; President, N. HATZIDAKIS; Vice-Presidents, G. REMOUNDOS and P. ZERVOS; Secretaries, N. SAKELLARIOS and A. ARVANITIS; Treasurer, G. AUTONOPOULOS. We have referred elsewhere in this issue to the Society's *Bulletin* [1920, 314].

We have already referred to the plan for making the University of Strasbourg next to Paris the greatest center in France for mathematical study [*1919*, 371]. The research courses for 1920–21 leading to the doctorate have been anonunced. For the first semester (November 1, 1920–February 28, 1921) they are in mathe-

matical physics by Professor BAUER, and in higher analysis by Professor FRÉCHET. During the second semester (March 1–June 30, 1921) these professors will continue their lectures, and in addition, Professor VILLAT will lecture on hydrodynamics, Professor PÉRÈS on differential geometry, Professor VALIRON on theory of functions. The degree which would be normally sought by the American student at Strasbourg is the "doctorat de l'Université de Strasbourg." The higher (state) degree, "doctorat ès sciences mathématiques," first granted in 1811, is the one almost invariably sought by the Frenchman. Only one American, Miss DOROTHEA KLUMPE, has ever received this degree from a French university (1893). Professor HARRIS HANCOCK is the only American who has received the French degree of "doctorat de l'université" in mathematics (1903). Before 1870 sixteen persons received state doctorates at the University of Strasbourg. Among them was J. G. DOSTOR author of the well known *Eléments de la Théorie des Déterminants* (1877, second edition, 1883).

The late Moritz Cantor's library [see *1920*, 280] is offered for sale by a Leipzig firm for one hundred and twenty nine thousand six hundred marks. It contains about 2000 volumes and 2500 pamphlets.—Darboux's library has been purchased by Hermann of Paris.

THE WORK OF THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS

JUNE 10, 1920 [*1920*, 145–146]

The National Committee on Mathematical Requirements held a meeting in Chicago on April 23 and 24, 1920. The principal topic discussed at this meeting was the preliminary report on "Junior High School Mathematics" prepared for the Committee by Mr. J. A. Foberg. After detailed discussion and some amendment and revision, the report was adopted by the Committee and its publication as a preliminary report authorized. It will be published by the U. S. Bureau of Education as one of its secondary school circulars.

This report considers briefly the history of the Junior High School Movement, its purposes and organization; it proposes certain general principles to govern the organization of work in mathematics in the junior high school and considers in some detail the topics to be included. The Committee feels that much experimentation must still precede any attempt at standardization of the junior high school curriculum in mathematics. Its recommendations are intended to form the basis of study, discussion and class room experiment. To this end the coöperation of teachers and supervisors is earnestly solicited. The National Committee hopes to act as a clearing house for constructive criticism based on actual class room experience.

The following resolution was adopted:

Resolved: That the National Committee on Mathematical Requirements approves the junior high school form of organization and urges its general adoption in the conviction that it will secure greater efficiency in the teaching of mathematics.

Reports of progress were made by subcommittees on The Training of Teachers, Experimental Schools and Courses, Disciplinary Values and Transfer of Training, Elective Courses in Mathematics for High Schools, and Mental Tests. It is expected that preliminary reports on all of these topics will be ready for consideration by the Committee at its next meeting on September 2, 3, and 4. The attention of experimental schools throughout the country is called to the report on this subject being prepared for the Committee by Mr. Raleigh Schorling of the Lincoln School, New York City. Any experimental schools or schools giving experimental courses in mathematics who desire to be represented in this report should communicate with Mr. Schorling without delay, if they have not already done so. A subcommittee on the Standardization of Terminology and Symoblism with Professor D. E. Smith as chairman and a subcommittee on Junior College Mathematics with Mr. A. C. Olney as chairman were appointed J. W. Young, Raleigh Schorling, and W. F. Downey were authorized to take steps to initiate investigations into the mathematical elements entering into various industries, professions, vocations, etc.

A budget for the coming year based on the recent appropriation of the General Education Board of $25,000 for the use of the Committee in completing its work was adopted. It is hoped that the increase in the item allowed for traveling expenses in this budget will make it possible for representatives of the Committee to reach educational meetings in all sections of the country where such representatives are desired to discuss the various reports of the Committee. Nearly 70 organizations are at present actively coöperating with the Committee and it is hoped that many others will communicate with the chairman in the interest of furthering the nation-wide study and discussion which is already under way. J. W. Young, 24 Musgrove Building, Hanover, New Hampshire and J. A. Foberg, 3829 North Tripp Avenue, Chicago, Illinois were reëlected chairman and vice-chairman, respectively, of the Committee for the ensuing year.

J. W. YOUNG, *Chairman.*

NOTE.

Through failure of the mail in returning proof an error occured in the announcement of John Wiley and Sons in the May issue of this MONTHLY. This stands corrected in their announcement of the present issue.

THE MAY MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The seventh regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at Goucher College, Baltimore, Md., on May 15, 1920. The meeting consisted of two sessions with Professor Ralph E. Root presiding.

The attendance was forty, including the following twenty-eight members of the Association: O. S. Adams, J. J. Arnaud, H. G. Avers, Clara L. Bacon, Sarah Beall, A. A. Bennett, G. A. Bingley, J. A. Bullard, G. R. Clements, A. Cohen, J. B. Coleman, G. H. Cresse, A. Dillingham, J. B. Eppes, W. M. Hamilton, P. E. Hemke, L. S. Hulburt, W. D. Lambert, A. E. Landry, Florence P. Lewis, J. J. Luck, F. D. Murnaghan, C. E. Norwood, L. J. Reed, R. E. Root, J. E. Rowe, T. McN. Simpson, Jr., G. F. Winslow, Jr.

A generous luncheon was served to the members and their guests by Goucher College. A committee to coöperate with the National Committee on Mathematical Requirements was appointed, consisting of Professor Abraham Cohen, chairman, Professor Clara L. Bacon, and Mr. Harry English. Professor L. S. HULBURT was elected chairman of the Section, O. S. ADAMS, secretary-treasurer and Professor G. R. CLEMENTS, member of the executive committee. The next meeting will be held at Annapolis, probably early in December.

The following papers were read:

(1) "Determination of longitude by the U. S. Coast and Geodetic Survey" (Illustrated) by Miss SARAH BEALL, Coast and Geodetic Survey;

(2) "Graphical methods applied to a curve of pursuit problem" by Mr. F. V. MORLEY, Johns Hopkins University;

(3) "Modular geometry" by Professor A. A. BENNETT, Technical Staff, Army Ordnance, Baltimore, Md.;

(4) "Some intrinsic geometric properties of plane curves and the related algebraic functions" by Professor J. B. COLEMAN, University of South Carolina;

(5) "Coördinate systems in modern ballistics" by Capt. R. S. HOAR, Aberdeen Proving Grounds.

(6) "Plane curves developed upon cylinders and cones" by Professor J. J. LUCK, University of Virginia;

(7) "Some maxima and minima in elementary geometry" by Dr. T. H. GRONWALL, Technical Staff, Army Ordnance;

(8) "The mathematical basis of the Einstein theory of relativity" by Dr. F. D. MURNAGHAN, Johns Hopkins University;

(9) "Singular curves of a plane pencil" by Professor C. C. BRAMBLE, U. S. Naval Academy;

(10) "An expression for the tangential velocity of a particle" by Dr. G. H. CRESSE, U. S. Naval Academy;